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AN APPLICATION OF THE PSEUDO-SPHERE TO THE INFORMATION GEOMETRY

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In the present paper we study some statistical parameter spaces which have a special role in the information geometry. Both analytical and geometrical representations of each of them are found out. The rotational surfaces of constant negative Gauss curvature are also considered and parametrically represented. We prove that the pseudo-sphere is actually a particular case of all of those surfaces. The pseuso-sphere is used for characterizing the statistical parameter spaces constituted by the normal, the exponential and the logistic density functions. A comparative representation of these three parameter spaces is given as well.

1. Introduction. The information geometry is mainly based on the study of the statistical parameter spaces. These spaces are sets of parameters which are characterized by some probability density functions, i.e. they are not supposed to have geometrical structure. However, as C. R. Rao proved in [1], the Fisher's information matrix satisfies all the properties of a Riemannian metric. Thus the statistical parameter spaces can be treated as Riemannian spaces. This fact acquires the special significance of the Riemannian geometry for the development of the information geometry. The Riemannian approach to the study of the statistical parameter spaces has been intensively used recently by many authors. Most of the important Riemannian characteristics of the main statistical parameter spaces have been already studied ([2]–[3]). However, there are still many open problems in the information geometry mostly concerning both the interpretation of some of these characteristics and the representation of the statistical parameter spaces. We considered some of those problems in our previous research ([4], [5]) and found their solutions.

In the present paper we continue our study of the statistical parameter spaces and characterize some of them. We consider the statistical parameter spaces constituted by three main probability density functions: the normal, the exponential and the logisitc one. All of them are spaces of constant negative Gauss curvature and have a special role in the information geometry. We also study the rotational surfaces of constant negative Gauss curvature G and apply them to obtain a representation of the statistical parameter spaces mentioned above. In section 2 we find a general analytical representation of all the surfaces of constant negative Gauss curvature. We prove that the pseudo-sphere which

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is the only well-known example of a rotational surface of constant negative Gauss curvature is actually a particular case of all the surfaces of $G < 0$ given by our parametrical representation. Using these results, in Section 3 we characterize the statistical parameter spaces of the normal, the exponential and the logistic probability density functions by finding both analytical and geometric representations for each of them. Moreover, we give a comparative geometrical representation of those three statistical parameter spaces all together.

2. Rotational surfaces of constant negative Gauss curvature . Let $f(u)$ and $g(u)$ be continuous and differentiable functions, u be a parameter, $u \in (\infty; -\infty)$, and $l:\ x = f(u), \quad z = g(u)$, be an arbitrary curve in the coordinate plane Oxz. The surface S obtained by a rotation of the curve l around the coordinate axis Oz has the following representation:

$$S:\ x = f(u)\cos v, \quad y = f(u)\sin v, \quad z = g(u), \tag{2.1}$$

where the parameter $v \in [0; 2\pi]$.

The Gauss curvature of a rotational surface is given by the formula ([6]):

$$G = -\frac{f''}{f}, \tag{2.2}$$

where $f'' = f''(u) = \dfrac{d^2 f(u)}{du^2}$ and ${f'}^2 + {g'}^2 = 1$.

In case the Gauss curvature G of the surface S is a constant c the differential equation (2.2) takes the form $cf + f'' = 0$. It is obvious that if $c = 0$, then we get $f(u) = C_1 u$, where C_1 is an integration constant. Independently on what kind the function $g(u)$ is of, we still have a surface of constant Gauss curvature $G = 0$. The plane is an example for a surface of constant zero Gauss curvature.

A well-known example for a rotational surface of constant positive Gauss curvature is the sphere. We know that the Gauss curvature of a sphere of radius R is $G = 1/R^2$ ([6]). A less known fact, however, is that there are two more kinds of surfaces of constant positive Gauss curvature. They have been only briefly mentioned in [6], but not enough studied there. Representations of all of these three kinds of surfaces of constant positive Gauss curvature were found in [4] (see also [5]).

Let us now pay attention on the rotational surfaces of constant negative Gauss curvature. They are very useful for illustrating of some important statistical parameter spaces. The only well-known example for a rotational surface of constant negative Gauss curvature is the pseudo-sphere. But besides of it there are two more kinds of rotational surfaces of constant negative Gauss curvature which have not yet been studied. Only a short notice of their existence can be found in [6]. They have been represented and for the first time applied to the Information geometry in [4]. In this section we give a general analytical expression for all these surfaces and prove that the pseudo-sphere is actually a particular case of all the surfaces which have constant negative Gauss curvature. Next we use these surfaces to give a representation of three important statistical parameter spaces.

We consider the rotational surfaces which have Gauss curvature $G = -c^2$, where c is an arbitrary constant. Then we get from (2.2):

$$c^2 f - f'' = 0. \tag{2.3}$$

The solution of the differential equation (2.3) is

$$f = C_1 \exp^{-cu} + C_2 \exp^{cu}, \tag{2.4}$$

where C_1, C_2 are integration constants. If $C_1 = C_2 = 0$, we get the trivial case $f(u) = 0$. Further we consider all cases except it. Depending on the constants C_1 and C_2 we have three kinds of rotational surfaces represented by (2.4) ([7]):

I kind. $C_1 = 0$ or $C_2 = 0$.

I. A). $C_1 \neq O, C_2 = 0$.

The functions $f(u)$ and $g(u)$ in this case take the forms:

$$f(u) = C_1 \exp^{-cu}, \qquad g(u) = \pm\frac{1}{c}\int_0^u \sqrt{1 - (-cC_1 \exp^{-ct})^2}\,dt. \tag{2.5}$$

For convenience we denote $C_1 = \alpha_1$, $c = \alpha_2$ and unify c and all the integration constants by α_i, $i = 1, 2, 3, 4$. Integrating (2.5), we get:

$$g(u) = \alpha_3 \left[\log\tan\frac{\arcsin\alpha_4 e^{-\alpha_2 u}}{2} + \cos(\arcsin\alpha_4 e^{-\alpha_2 u})\right]$$

where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (C_1, c, \mp\frac{1}{c}, C_1 c)$$

and

$$\alpha_4 e^{-\alpha_2 u} \leq 1 \quad , \quad 0 < \alpha_1 \leq \frac{\exp^{cu}}{c} \quad , \quad (0 \leq u \leq +\infty). \tag{2.6}$$

The functions $f(u)$ and $g(u)$ define a curve in the plane Oz given by

$$l: \quad x = \alpha_1 \exp^{-\alpha_2 u}, \quad z = \alpha_3 \left[\log\tan\frac{\arcsin\alpha_4 e^{-\alpha_2 u}}{2} + \cos(\arcsin\alpha_4 e^{-\alpha_2 u})\right] \tag{2.7}$$

and the conditions (2.6) are satisfied. In case $\alpha_1 = 1/c$ we get from (2.7) the parametric equations of the so-called *tractriss curve*:

$$l: \quad x = \frac{1}{c}\exp^{-cu}, \quad z = \mp\frac{1}{c}\left[\log\tan\frac{\arcsin e^{-cu}}{2} + \cos(\arcsin e^{-cu})\right]. \tag{2.8}$$

The surface obtained by the rotation of the tractriss curve (2.8) around the axis Oz is the pseudo-sphere. Due to (2.1) it can be represented by the following parametrical equations:

$$S: \begin{cases} x = \frac{1}{c}\exp^{-cu}\cos v \\ y = \frac{1}{c}\exp^{-cu}\sin v \\ z = \mp\frac{1}{c}\left[\log\tan\frac{\arcsin e^{-cu}}{2} + \cos(\arcsin e^{-cu})\right], \end{cases} \tag{2.9}$$

where $u \in [0; 2\pi]$ and $v \in [0; \pi]$.

I. B). $C_1 = 0$, $C_2 \neq 0$.

In this case we denote $C_2 = \alpha_1$ and after similar considerations as in the previous case I. A) we get the expressions for both functions $f(u)$ and $g(u)$. They are equivalent to (2.7) for $u \in (-\infty, 0]$, i.e. after the rotation of the curve l in this case we get the same preudo-sphere as in the previous case. The conditions for the constant α_1, however, are $0 \leq \alpha_1 \leq \exp^{-cu}/c$.

Thus, we obtained that the preudo-sphere is actually a particular case of all the rotational surfaces of constant negative Gauss curvature represented by (2.4). We can get it from (2.4) in case one of the integration constants C_1 or C_2 is zero. If another constant is equal to $1/c$, then the pseudo-sphere obtained has constant $1/c$.

Let us now consider the next case: neither of the integration constants C_1 and C_2 is zero. The surfaces obtained in such a case so that the integral in (2.5) to be calculated can be represented in the following two groups: $\{C_1 = -C_2\}$ and $\{C_1 = C_2\}$, i.e. we have another two kinds of rotational surfaces of constant negative Gauss curvature.

II kind. $C_1 = -C_2$.

In this case the rotational surface has the following parametrical representation:

$$S : \begin{cases} x = \beta_1 \sinh \beta_2 u \cos v \\ y = \beta_1 \sinh \beta_2 u \sin v \\ z = i\beta_3 E(i\beta_2 u \mid -\beta_4), \end{cases} \tag{2.10}$$

where $E(i\beta_2 u \mid -\beta_4)$ is an extended elliptic integral of second kind and

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (\beta_1, c, \mp\sqrt{\frac{1}{c^2} - {\beta_1}^2}, \frac{\beta_1^2 c^2}{1 - \beta_1^2 c^2}),$$

$$-\frac{1}{c}\operatorname{arc\,sinh}\sqrt{\frac{1}{\beta_4}} \leq u \leq \frac{1}{c}\operatorname{arc\,sinh}\sqrt{\frac{1}{\beta_4}}\ , \qquad 0 < \beta_1 < \frac{1}{c}.$$

III kind. $C_1 = C_2$.

The surface of this kind is parametrically represented as follows:

$$S : \begin{cases} x = \gamma_1 \cosh \gamma_2 u \cos v \\ y = \gamma_1 \cosh \gamma_2 u \sin v \\ z = i\gamma_3 E(i\gamma_2 u \mid -\gamma_4), \end{cases} \tag{2.11}$$

where

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\gamma_1, c, \mp\frac{1}{c}, \gamma_1^2 c^2) \quad \text{and} \quad -\frac{1}{c}\operatorname{arcsinh}\sqrt{\frac{1}{\gamma_4}} \leq u \leq \frac{1}{c}\operatorname{arcsinh}\sqrt{\frac{1}{\gamma_4}}.$$

Both kinds of surfaces defined by (2.10) and (2.11), respectively, are illustrated in [4] and [5].

In the next section we use the rotational surface of I-st kind, i.e. the pseudo-sphere for describing of some statistical parameter spaces which have a special role in the Information geometry.

3. Representations of three main statistical parameter spaces.

3.1. Statistical parameter space constituted by the normal probability

density function. The statistical parameter space constituted by the normal probability density function is one of the fundamental and most popular statistical parameter spaces in Information geometry. It has been already studied ([2], [3]) and most of its characteristics are well-known. The problem to find a geometrical representation of this parameter space has not been solved yet. In the present section we give both a parametric and a geometrical representation of the space constituted by the normal probability density function.

Definition 1. *The probability density function in case of a 1-dimensional normal distribution* is defined in the following way:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{3.1}$$

where x is a random variable, μ is the mean value, σ is the standard deviation and σ^2 is the variance.

The statistical parameter space N constituted by the normal probability density function (3.1) is a space of constant negative Gauss curvature:

$$G_N = -\frac{1}{2}. \tag{3.2}$$

Using (3.2) and (2.9) for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\sqrt{2}, \frac{\sqrt{2}}{2}, -\sqrt{2}, 1)$, we can analytically represent the space N in the following way:

$$S_N : \begin{cases} x = \sqrt{2}\exp^{-\frac{\sqrt{2}}{2}u} \cos v \\ y = \sqrt{2}\exp^{-\frac{\sqrt{2}}{2}u} \sin v \\ z = -\sqrt{2}\left[\log\tan\dfrac{\arcsin e^{-\frac{\sqrt{2}}{2}u}}{2} + \cos(\arcsin e^{-\frac{\sqrt{2}}{2}u})\right]. \end{cases} \tag{3.3}$$

The rotational surface S_N given by (3.3) for $v \in [\pi/2; \pi]$ is represented on Fig. 1.

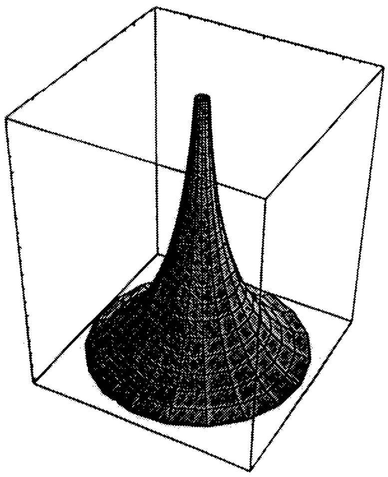


Fig. 1. The parameter space N.

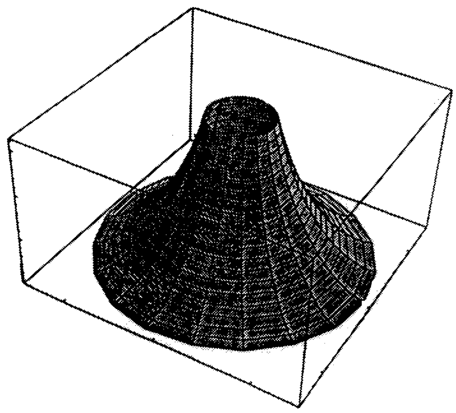


Fig. 2. The parameter space E.

3.2. Statistical parameter space constituted by the exponential probability density function.

Definition 2. *The exponential probability density function* is defined by the equation:

$$f(x;\mu,\sigma)=\frac{1}{\sigma}e^{\frac{-(x-\mu)}{\sigma}}, \tag{3.5}$$

where x $(x>\mu)$ is a random variable, μ is the mean value, σ is the standard deviation.

We consider the statistical parameter space constituted by the exponential probability density function (3.5) and note it by E. The Gauss curvature of the parameter space E is ([2])

$$G_E=-1, \tag{3.5}$$

Then having in mind (3.5) and (2.9) for $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(1,1,-1,1)$, we get the following analytical representation of the statistical parameter space E constituted by the exponential probability density function:

$$S_E:\begin{cases} x=\exp^{-u}\cos v\\ y=\exp^{-u}\sin v\\ z=-\left[\log\tan\dfrac{\arcsin e^{-u}}{2}+\cos(\arcsin e^{-u})\right].\end{cases} \tag{3.6}$$

The rotational surface S_E defined by (3.6) for $v\in[\pi/2;\pi]$ is represented on Fig. 2.

3.3. The statistical parameter space constituted by the logisitc density function. The logisitc function has some properties which make it very suitable for creating mathematical models of some biological phenomena. The logisitc function has been used also in some mathematical models of the neural network which has been widely investigated recently. The logisitc distribution has been applied to the model of the growth of the human beings population. It has been already used also as an approximation of the normal distribution. In [8] we constituted a statistical parameter space using the logisitc density function. We studied all the most important from an information point of view Riemannian characteristics of this statistical parameter space ([4]). Now we give a representation of the parameter space L.

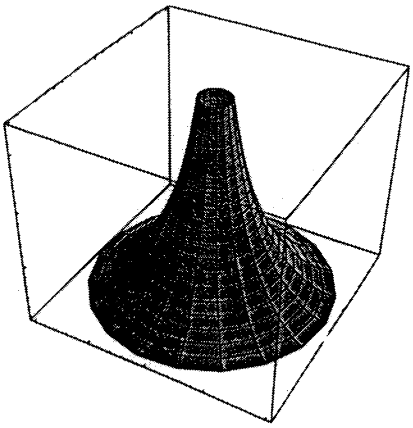
Definition 7. *The logistic function $F(x;a,b)$ and the logistic density function $f(x;a,b)$* are given by the equations:

$$F(x;a,b)=\frac{1}{1+e^{ax-b}},\qquad \frac{dF}{dx}\overset{def}{=}f(x;a,b)=\frac{-ae^{ax-b}}{(1+e^{ax-b})^2},\quad a<0, \tag{3.7}$$

respectively, where x is a random variable and a, b are parameters.

Let L be the statistical parameter space constituted by the logistic density function (3.7). In [4], we studied the parameter space L and obtained its characteristics. We also proved that the statistical parameter space constituted by the logistic density function is a space of constant negative Gauss curvature

$$G_L=-\frac{9}{\pi^2+3}. \tag{3.8}$$



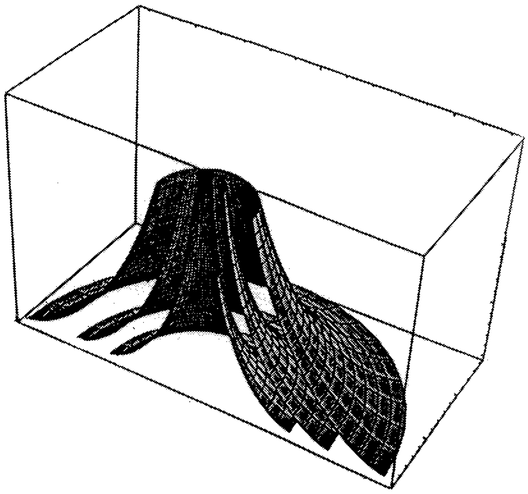


Fig. 3. The parameter space L. Fig. 4. The parameter spaces E, L and N.

The parameter space L as well as the parameter spaces N and E, can be represented by a pseudo-sphere (Fig. 3). The constant of this pseudo-sphere is $3/\sqrt{\pi^2+3}$.

Let us now consider all three statistical parameter spaces represented above. Taking into account (3.2), (3.5) and (3.8), we can notice an interesting fact. The Gauss curvatures of the statistical parameter spaces N, E and L constituted by the normal, the exponential and the logistic density functions, respectively, satisfy the following relation:

$$G_E < G_L < G_N.$$

We represent those three parameter spaces all together on Fig. 4.

Remark. All the rotational surfaces representing the parameter spaces N, E and L on Figures 1-4 are drown for $v \in [\pi/2; \pi]$. In case $v \in [0; \pi]$ we get also another part of each pseudo-sphere which is symmetrical to the represented one in respect to the coordinate plane Oxy. We also notice that when the parameter $v \to 0$ and $v \to \pi$ the z-coordinates of the pseudo-spheres go to $-\infty$ and $+\infty$, respectively. We have $z \in [0; 2]$ on Figures 1, 3, 4 and $z \in [0; 1.5]$ on Figure 2.

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ЕДНО ПРИЛОЖЕНИЕ НА ПСЕВДО-СФЕРАТА В ИНФОРМАЦИОННАТА ГЕОМЕТРИЯ

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В настоящата работа разглеждаме някои статистически параметрични пространства, които имат специална роля в информационната геометрия. Намерени са както аналитично, така и геометрично представяне за всяко от тях. Разгледани са също и ротационните повърхнини с постоянна отрицателна Гаусова кривина, за която е намерено и параметрично представяне. Намерено е едно приложение на псевдо-сферата за характеризиране на статистическите параметрични пространства, определени от нормалната, експоненциалната и логистичната функция на гъстота. Дадено е и сравнително представяне на посочените три параметрични пространства.