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SOME NECESSARY AND SOME SUFFICIENT CONDITIONS ABOUT THE 3-SATISFIABILITY PROBLEM

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The so called 3-satisfiability problem takes an important place in the theory of NP-completeness of problems and algorithms. This work is a continuation of a previous one, where conjunctive normal forms of the zero function namely forms, such that any disjunct of which has no more than three literals (3-CNForms) have been considered and classified. Here we shall deduce some necessary and some sufficient conditions which determine whether given sets of 3-CNForms are or are not forms of the zero function. These conditions are polynomial-time verifiable. Some estimations of the number of these forms are also given.

1. Introduction and preliminary results. Each function from the set $\mathcal{F}_2^n = \{f | f : \{0,1\}^n \to \{0,1\}\}$ is called a *Boolean function* of n independent variables $x_1, x_2, \ldots, x_n$ and it is denoted $f(x_1, x_2, \ldots, x_n)$. The union $\mathcal{F}_2 = \mathcal{F}_2^1 \cup \mathcal{F}_2^2 \cup \ldots \cup \mathcal{F}_2^n \cup \ldots$ denotes the set of all Boolean functions. The function $f_0(x_1, x_2, \ldots, x_n)$, where $f_0(a_1, a_2, \ldots, a_n) = 0$ for any true assignment $(a_1, a_2, \ldots, a_n)$ from the n-dimensional Boolean cube $\{0,1\}^n$ (for $n = 1, 2, \ldots$) is called a *zero function* and denoted $\tilde{0}$. The famous theorem of George Bool proves that for any function $f \in \mathcal{F}_2$ there exists a formula over the set $\{\bar{x}, x_1 \vee x_2, x_1 x_2\}$.

The function $f(x,d) = x^d$ is defined by $x^d = \begin{cases} \bar{x} & if\, d = 0 \\ x & if\, d = 1 \end{cases}$ and the formula $x_{i_1}^{d_1} \vee \vee x_{i_2}^{d_2} \vee \ldots \vee x_{i_m}^{d_m}$ where $x_{i_j} \neq x_{i_k}$ for $j \neq k$ is called a *disjunct of length* m. We consider the zero function as a disjunct of length 0. The formula $D_1 \wedge D_2 \wedge \cdots \wedge D_r$, where $D_1, D_2, \ldots, D_r$ are different disjuncts is called a *conjunctive normal form* (CNForm) of the corresponding function. The Bool's theorem implies that at least one CNForm exists for any Boolean function different from $\tilde{1}$. The true assignment $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ satisfies a formula φ of the Boolean function f_φ of n variables if $f_\varphi(a_1, a_2, \ldots, a_n) = 1$.

A problem of recognition from the theory of Boolean functions has the honor to be the first member of the whole class of interesting problems for whose solving no good algorithms are known – the so called *NP-complete problems* [1]. This problem is called *CNForm Satisfiability* (or simply *Satisfiability*) and it is formulated so: "A Boolean function in a conjunctive normal form φ is given. Is there a true assignment that satisfies φ?" The basic theorem of Cook in [1] states that the Satisfiability is NP-complete problem. When each of the disjuncts in the CNForm $D_1 \wedge D_2 \wedge \cdots \wedge D_r$ has a length = 3, this form is called a *3-CNForm.* It is also proved that the Satisfiability problem is reducible in polynomial time to the following problem, called a *3-Satisfiability*: "A Boolean function f_φ in a 3-CNForm φ is given. Is there a true assignment that satisfies φ?". That is why we shall consider only 3-CNForms. The question of the 3-Satisfiability problem can be formulated as following: "Whether the 3-CNForm φ is a form of the zero function?".

In [2] we started to examine the 3-CNForms of $\tilde{0}$ with up to 3 essential variables under some "equivalencies" and gave a full classification under these equivalencies. The basic definitions and assertions are:

Definition. *The 3-CNForm φ' of $\tilde{0}$ is pn-equivalent to the formula φ of $\tilde{0}$ if it is obtained from φ by a permutation and/or negations of variables.*

Definition. *Let $D_1 \wedge D_2 \wedge \cdots \wedge D_r$ be a 3-CNForm φ of $\tilde{0}$. For any $j \in N, 1 \le j \le r$ the form $D_1 \wedge \cdots \wedge D_{j-1} \wedge D_{j+1} \wedge \cdots \wedge D_r$ is called a cut of D_j in φ. The form φ of $\tilde{0}$ is called cut-irreducible if any cut in φ makes it a 3-CNForm which is not a form of $\tilde{0}$.*

Definition. *Let $D_1 \wedge D_2 \wedge \cdots \wedge D_r$ be a 3-CNForm of $\tilde{0}$. If $D_j = x_{j_1}^{\sigma_1} \vee x_{j_2}^{\sigma_2} \vee \ldots \vee x_{j_s}^{\sigma_s}$, $2 \le s \le n$ and $D'_j = x_{j_1}^{\sigma_1} \vee \ldots \vee x_{j_{k-1}}^{\sigma_{k-1}} \vee x_{j_{k+1}}^{\sigma_{k+1}} \vee \ldots \vee x_{j_s}^{\sigma_s}$ for some k, $1 \le k \le s$, then $D_1 \wedge \cdots \wedge D_{j-1} \wedge D'_j \wedge D_{j+1} \wedge \cdots \wedge D_r$ is called a puncture of the variable $x_{j_k}^{\sigma_k}$ in D_j. The form φ of $\tilde{0}$ is called puncture-irreducible if any puncture in φ reduces it to CNForm which is not cut-irreducible.*

Definition. *The 3-CNForm φ of $\tilde{0}$ is called irreducible if it is cut-irreducible and it is puncture-irreducible.*

For any function $f \in \mathcal{F}_2^n$ we shall denote

$$Z_f = \{(a_1, a_2, \ldots, a_n) | (a_1, a_2, \ldots, a_n) \in \{0,1\}^n, f(a_1, a_2, \ldots, a_n) = 0\}.$$

Lemma 1.1. *If the disjunct $D = x_{i_1}^{\sigma_1} \vee x_{i_2}^{\sigma_2} \vee \ldots \vee x_{i_k}^{\sigma_k}$, $0 \le k \le n$, is a Boolean function of n variables, then Z_D is a $(n-k)$-dimensional subcube of the n-dimensional cube $\{0,1\}^n$ and so $|Z_D| = 2^{n-k}$.*

Lemma 1.2. *Let the disjuncts D_1 and D_2 be Boolean functions of n variables. If there is x_i in D_1 and $\bar{x}_i$ in D_2, then $Z_{D_1} \cap Z_{D_2} = \emptyset$.*

Lemma 1.3. *If the disjuncts $D_1, D_2, \ldots, D_k$ of length 1 are Boolean functions of n variables, different disjuncts comprise different variables and $\varphi = D_1 \wedge D_2 \wedge \cdots \wedge D_k$, then $|Z_\varphi| = 2^n - 2^{n-k}$.*

Lemma 1.4. *Let the disjuncts $D_1 = x_{i_1}^{a_1} \vee \ldots \vee x_{i_k}^{a_k} \vee x_{j_1}^{b_1} \vee \ldots \vee x_{j_p}^{b_p}$ and $D_2 = x_{i_1}^{a_1} \vee \ldots \vee x_{i_k}^{a_k} \vee x_{m_1}^{c_1} \vee \ldots \vee x_{m_q}^{c_q}$ are Boolean functions of n variables, $0 \le k \le n$, $0 \le p \le n-k$, $0 \le q \le n-k-p$ and $x_{j_s} \ne x_{m_t}$ for $s \ne t$. Then:*
1) $|Z_{D_1} \cap Z_{D_2}| = 2^{n-(k+p+q)}$
2) $|Z_{D_1 \wedge D_2}| = 2^{n-(k+p+q)}(2^p + 2^q - 1)$

Theorem 1.1. *The irreducible 3-CNForms $D_1 \wedge D_2 \wedge \cdots \wedge D_r$ of $\tilde{0}$ with up to 3 variables are:*

1) *for $r = 1$: the disjunct of length 0 or $\tilde{0}$;*
2) *for $r = 2$: $x_i \bar{x}_i$;*
3) *for $r = 3$: $x_i x_j (\bar{x}_i \vee \bar{x}_j)$;*
4) *for $r = 4$:*

— $x_i x_j x_k (\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
— $x_i x_j (\bar{x}_i \vee x_k)(\bar{x}_j \vee \bar{x}_k)$;
— $(x_i \vee x_j)(x_i \vee \bar{x}_j)(\bar{x}_i \vee x_j)(\bar{x}_i \vee \bar{x}_j)$;
5) *for* $r = 5$:
— $x_i(x_j \vee x_k)(x_j \vee \bar{x}_k)(\bar{x}_j \vee x_k)(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
— $(x_i \vee x_j)(x_i \vee x_k)(\bar{x}_i \vee x_j)(\bar{x}_i \vee x_k)(\bar{x}_j \vee \bar{x}_k)$;
— $(x_i \vee x_j)(x_i \vee x_k)(\bar{x}_i \vee x_j)(\bar{x}_i \vee \bar{x}_j)(\bar{x}_j \vee \bar{x}_k)$;
6) *for* $r = 6$:
— $(x_i \vee x_j)(x_i \vee x_k)(x_j \vee x_k)(\bar{x}_i \vee \bar{x}_j)(\bar{x}_i \vee \bar{x}_k)(\bar{x}_j \vee \bar{x}_k)$;
— $(x_i \vee x_j)(x_i \vee x_k)(\bar{x}_i \vee x_j)(\bar{x}_i \vee x_k)(x_i \vee \bar{x}_j \vee \bar{x}_k)(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
— $(x_i \vee x_j)(x_i \vee x_k)(\bar{x}_i \vee x_j)(\bar{x}_i \vee \bar{x}_k)(x_i \vee \bar{x}_j \vee \bar{x}_k)(\bar{x}_i \vee \bar{x}_j \vee x_k)$;
7) *for* $r = 7$:
— $(x_i \vee x_j)(x_i \vee x_k)(x_j \vee x_k)(x_i \vee \bar{x}_j \vee \bar{x}_k)(\bar{x}_i \vee x_j \vee \bar{x}_k)(\bar{x}_i \vee \bar{x}_j \vee x_k)(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
8) *for* $r = 8$:
— $(x_i \vee x_j \vee x_k)(x_i \vee x_j \vee \bar{x}_k)(x_i \vee \bar{x}_j \vee x_k)(x_i \vee \bar{x}_j \vee \bar{x}_k) \wedge$
$\wedge(\bar{x}_i \vee x_j \vee x_k)(\bar{x}_i \vee x_j \vee \bar{x}_k)(\bar{x}_i \vee \bar{x}_j \vee x_k)(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$.

2. Necessary and sufficient conditions in the 3-Satisfiability problem. Next we shall consider the 3-Satisfiability problem only in its original form, i.e. when each of the disjuncts in the 3-CNForms has a length exactly equal to 3. We suppose that there are no repetitions of disjuncts in the 3-CNForms and we shall consider them as boolean functions of so many variables as many they are in the corresponding form, i.e. of n essential variables in the general case. Our goal is to define some necessary and sufficient conditions, so that a 3-CNForm to be a form of $\tilde{0}$ and these conditions to be polynomial-time verifiable.

First we shall partition the set of all possible disjuncts of n variables into 4 non intersecting each other subsets accotding to the number of negations which they contain. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables. We shall note:

$$S_0 = \{x_i \vee x_j \vee x_k |\ x_i, x_j, x_k \in X\}. \text{ Therefore } |S_0| = \binom{3}{0}\binom{n}{3} = \binom{n}{3};$$

$$S_1 = \{\bar{x}_i \vee x_j \vee x_k |\ x_i, x_j, x_k \in X\}. \text{ Therefore } |S_1| = \binom{3}{1}\binom{n}{3} = 3\binom{n}{3};$$

$$S_2 = \{\bar{x}_i \vee \bar{x}_j \vee x_k |\ x_i, x_j, x_k \in X\}. \text{ Therefore } |S_2| = \binom{3}{2}\binom{n}{3} = 3\binom{n}{3};$$

$$S_3 = \{\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k |\ x_i, x_j, x_k \in X\}. \text{ Therefore } |S_3| = \binom{3}{3}\binom{n}{3} = \binom{n}{3}.$$

So the set of all possible such disjuncts of n variables is $S = S_0 \cup S_1 \cup S_2 \cup S_3$ and it consists generally of $8\binom{n}{3}$ disjuncts. Therefore $8\binom{n}{3}$ is the maximum length of the input of the 3-Satisfiability problem. We shall set $m = \binom{n}{3}$ and we shall use m only for this shorter notation further.

Definition. *The disjunct* $D = x_i^{d_i} \vee x_j^{d_j} \vee x_k^{d_k}$ covers *the vector* $\alpha = (a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ *if* $\alpha \in Z_D$, *i.e.* $D(a_1, a_2, \ldots, a_n) = 0$.

Now let us consider how many vectors $\alpha \in \{0,1\}^n$ are covered generally by the disjuncts from the sets $S_0, \ldots, S_3$. The disjuncts from the set S_0 cover all the vectors α, so that $0 \leq ||\alpha|| \leq n-3$, i.e. all the vectors which contain at least 3 zeros. Their number is $2^n - 1 - \frac{n(n+1)}{2}$. By analogy the disjucts from the set S_3 cover all the vectors

α, so that $3 \le ||\alpha|| \le n$, i.e. all the vectors which contain at least 3 ones, and so their number is $2^n - 1 - \frac{n(n+1)}{2}$ again. The disjuncts from the set S_1 cover all the vectors α, so that $1 \le ||\alpha|| \le n-2$, i.e. all the vectors which contain at least 2 zeros and at least 1 one. Their number is $2^n - 2 - n$. And by analogy the disjucts from the set S_2 cover all the vectors α, so that $2 \le ||\alpha|| \le n-1$, i.e. all the vectors which contain at least 1 zero and at least 2 ones, and so their number is $2^n - 2 - n$ again. A partition of the vectors from $\{0,1\}^n$ in accordance with their weight and which vectors each of the sets $S_0, \dots, S_3$ covers is shown on the next figure.

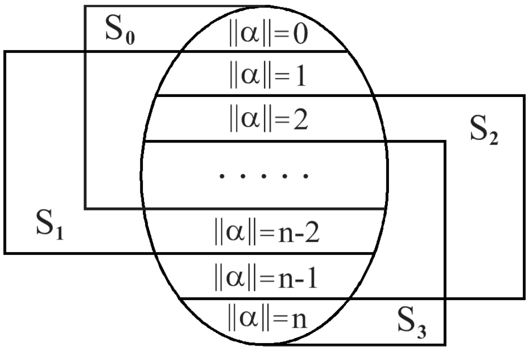

Fig. 1. Covering of the $n-$dimensional Boolean cube by the sets $S_0, \dots, S_3$.

The next table presents the number of vectors in $S_0, \dots, S_3$ for the first several values of n.

A set and a formula for its cardinality	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\lvert S_0\rvert = 2^n - 1 - \frac{n(n+1)}{2}$	1	5	16	42	99
$\lvert S_1\rvert = 2^n - 2 - n$	3	10	25	56	119
$\lvert S_2\rvert = 2^n - 2 - n$	3	10	25	56	119
$\lvert S_3\rvert = 2^n - 1 - \frac{n(n+1)}{2}$	1	5	16	42	99
$\lvert S_0 \cup S_1 \cup S_2 \cup S_3\rvert = 2^{n+2} - 6 - n(n+3)$	8	30	82	196	436
$\lvert\{0,1\}^n\rvert = 2^n$	8	16	32	64	128

Table 1. Cardinality of the sets $S_0, \dots, S_3$ and $\{0,1\}^n$ for $n = 3, \dots, 7$.

So, we can already formulate:

2.1. Necessary conditions for a 3-CNForm to be a form of $\tilde{0}$.

1. The number of the disjuncts in the 3-CNForm (i.e. the length of the form) must be at least 8.

Forms of $\tilde{0}$ with less than 8 disjuncts does not exist – in accordance with Theorem 1.1, case $r = 8$. So, if the input length of the 3-Satisfiability problem is less than 8 then the corresponding 3-CNForm is not a form of $\tilde{0}$. The number of these forms is $\sum_{i=0}^{7} \binom{8m}{i}$.

Definition. *The 3-CNForm of the type* $(x_i \vee x_j \vee x_k)(x_i \vee x_j \vee \bar{x}_k)(x_i \vee \bar{x}_j \vee x_k)(x_i \vee \bar{x}_j \vee \vee \bar{x}_k)(\bar{x}_i \vee x_j \vee x_k)(\bar{x}_i \vee x_j \vee \bar{x}_k)(\bar{x}_i \vee \bar{x}_j \vee x_k)(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$ *we shall call a minimal basic form (MBForm) of* $\tilde{0}$ *and we shall denote it with* $M_{i,j,k}$.

The next table shows the set S of all possible disjunct of the type $D = x_i^{\sigma_1} \vee x_j^{\sigma_2} \vee x_k^{\sigma_3}$, its partition in subsets $S_0, \dots, S_3$ and also its partition in MBForms.

Values of $(\sigma_1, \sigma_2, \sigma_3)$	Possible values of (i, j, k) (1, 2, 3)	$\cdots$	(1, 2, n)	$\cdots$	(i, j, k)	$\cdots$	(n-2, n-1, n)	Set
(0,0,0)	$\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$		$\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_n$		$\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k$		$\bar{x}_{n-2} \vee \bar{x}_{n-1} \vee \bar{x}_n$	S_3
(0,0,1)	$\bar{x}_1 \vee \bar{x}_2 \vee x_3$		$\bar{x}_1 \vee \bar{x}_2 \vee x_n$		$\bar{x}_i \vee \bar{x}_j \vee x_k$		$\bar{x}_{n-2} \vee \bar{x}_{n-1} \vee x_n$	
(0,1,0)	$\bar{x}_1 \vee x_2 \vee \bar{x}_3$	$\cdots$	$\bar{x}_1 \vee x_2 \vee \bar{x}_n$	$\cdots$	$\bar{x}_i \vee x_j \vee \bar{x}_k$	$\cdots$	$\bar{x}_{n-2} \vee x_{n-1} \vee \bar{x}_n$	S_2
(1,0,0)	$x_1 \vee \bar{x}_2 \vee \bar{x}_3$		$x_1 \vee \bar{x}_2 \vee \bar{x}_n$		$x_i \vee \bar{x}_j \vee \bar{x}_k$		$x_{n-2} \vee \bar{x}_{n-1} \vee \bar{x}_n$	
(0,1,1)	$\bar{x}_1 \vee x_2 \vee x_3$		$\bar{x}_1 \vee x_2 \vee x_n$		$\bar{x}_i \vee x_j \vee x_k$		$\bar{x}_{n-2} \vee x_{n-1} \vee x_n$	
(1,0,1)	$x_1 \vee \bar{x}_2 \vee x_3$	$\cdots$	$x_1 \vee \bar{x}_2 \vee x_n$	$\cdots$	$x_i \vee \bar{x}_j \vee x_k$	$\cdots$	$x_{n-2} \vee \bar{x}_{n-1} \vee x_n$	S_1
(1,1,0)	$x_1 \vee x_2 \vee \bar{x}_3$		$x_1 \vee x_2 \vee \bar{x}_n$		$x_i \vee x_j \vee \bar{x}_k$		$x_{n-2} \vee x_{n-1} \vee \bar{x}_n$	
(1,1,1)	$x_1 \vee x_2 \vee x_3$		$x_1 \vee x_2 \vee x_n$		$x_i \vee x_j \vee x_k$		$x_{n-2} \vee x_{n-1} \vee x_n$	S_0
MBForms	$M_{1,2,3}$	$\cdots$	$M_{1,2,n}$	$\cdots$	$M_{i,j,k}$	$\cdots$	$M_{n-2,n-1,n}$	

Table 2. The set S and its partitions.

2. *An arbitrary 3-CNForm of* $\tilde{0}$ *must have at least a disjunct from the set* S_0 (otherwise the vector $(0, 0, \dots, 0)$ remains uncovered) *and at least a disjunct from the set* S_3 (otherwise the vector $(1, 1, \dots, 1)$ remains uncovered) - see Figure 1.

How many are the 3-CNForms which does not contain a disjunct from at least one of the sets S_0 or S_3, i.e. which are not forms of $\tilde{0}$? These, which does not contain a disjunct from S_0 are 2^{7m} and so many are these which does not contain a disjunct from S_3. By the principle of inclusion and exclusion we obtain $2^{7m} + 2^{7m} - 2^{6m} = 2^{7m+1} - 2^{6m}$ forms generally. Amongst them $\sum_{i=0}^{7} \binom{6m}{i}$ contain less than 8 disjuncts. So the number of 3-CNForms which are not forms of $\tilde{0}$ is at least

$$2^{7m+1} - 2^{6m} + \sum_{i=0}^{7} \binom{8m}{i} - \sum_{i=0}^{7} \binom{6m}{i}.$$

This is quite few: $\approx 2^{1-m}$ as a part of all possible 3-CNForms.

3. *In dependence of the number of variables in an arbitrary 3-CNForm of* $\tilde{0}$ *it is necessary to participate:*

— *in the case of 3 variables – all the disjuncts from the sets* $S_0, \dots, S_3$*;*

— *in the case of 4 variables – disjuncts at least from 3 of these sets;*

— *in the case of 5 and more variables – disjuncts at least from 2 of these sets (when the sets are exactly 2 they must be* S_0 *and* S_3*).*

This assertion follows from Theorem 1.1, Table 1 and Figure 1.

2.2. Sufficient conditions a 3-CNForm to be a form of $\tilde{0}$.

1. *Each 3-CNForm which contains all the disjuncts of an arbitrary MBForm (as a subset of its disjuncts) is a form of* $\tilde{0}$. The truth of this assertion follows directly from Theorem 1.1, case $r = 8$. Table 2 shows a partition of the set of all possible disjuncts S in m different MBForms. We shall estimate the number of 3-CNForms of $\tilde{0}$ from below, using this sufficient condition. Let A denote the set of all 3-CNForms which contain (as a subset) all the disjuncts of MBForm $M_{1,2,3}$ and with B – the set of all 3-CNForms which contain (as a subset) all the disjuncts of $M_{1,2,4}$. So $|A| = |B| = 2^{8m-8}$. The

union $A \cup B$ will consist of all possible 3-CNForms which contain (as a subset) all the disjuncts of an arbitrary MBForm (or forms) and therefore their number is $|A \cup B| = |A| + |B| - |A \cap B| = 2^{8m-8} + 2^{8m-8} - 2^{8m-16} = 2^{8m}(2^{-7} - 2^{-16})$. This number is $\approx \dfrac{1}{128}$ the part of all possible 3-CNForms. It is interesting that this correlation does not depend on n.

2. Each 3-CNForm, which contains more than $7m$ disjuncts is a form of $\tilde{0}$. To prove this assertion we shall note, that a 3-CNForm which consists of all possible $8m$ disjuncts obviously is a form of $\tilde{0}$. What is the maximal number of disjucts which we can remove so that the obtained form to stay a form of $\tilde{0}$? To answer this question we shall consider how many disjuncts cover one and the same vector $\alpha \in \{0,1\}^n$. Let $||\alpha|| = k, 0 \le k \le n$ and α has ones in positions $i_1, i_2, \ldots, i_k$. We form the set of variables $Y = \{\bar{x}_{i_1}, \bar{x}_{i_2}, \ldots, \bar{x}_{i_k}, x_{j_1}, x_{j_2}, \ldots, x_{j_{n-k}}\}$ and we consider all possible disjuncts $D = y_p \vee y_q \vee y_r$ where $y_p, y_q, y_r \in Y$. Obviously $D(\alpha) = 0$. Therefore the number of all possible such disjuncts is equal to the number of ways in which we can choose 3 variables y_p, y_q, y_r from Y, i.e. equal to m. So if we remove $1, 2, 3, \ldots, m-1$ arbitrary disjuncts from a 3-CNForm which consists of all possible $8m$ disjuncts this form remains a form of $\tilde{0}$. In this way we obtain $\displaystyle\sum_{i=0}^{m-1} \binom{8m}{i}$ such forms. We shall note that all these forms are a subset of the 3-CNFforms of $\tilde{0}$ considered in sufficient condition 1 and the given proof and estimation can be deduced in another way.

2.3. The other cases. Now we shall consider the cases when we have necessary conditions but we have not sufficient conditions (unfortunately these necessary conditions are not sufficient conditions). Let us have a 3-CNForm of the function $f \in \mathcal{F}_2^n$, $f(x_1, x_2, \ldots, x_n) = D_1 \wedge D_2 \wedge \ldots \wedge D_r$, $8 \le r \le 7m$. Let also this form have at least one disjunct from the set S_0 and at least one disjunct from the set S_3. We do not know a polynomial-time verifiable criterion whether this form is a form of $\tilde{0}$, i.e. whether $f(x_1, x_2, \ldots, x_n) = \tilde{0}$. Following the given definitions and assertions we can say that Lemma 1.4 is a particular case of the principle of inclusion and exclusion − for 2 disjuncts or more exactly for their zero sets. When $r > 2$ we shall consider the coverings of the sets $Z_{D_1}, Z_{D_2}, \ldots, Z_{D_r}$ and we shall apply the same principle, i.e.:

(1) $$|Z_f| = |Z_{D_1 \wedge D_2 \wedge \ldots \wedge D_r}| = |Z_{D_1} \cup Z_{D_2} \cup \ldots \cup Z_{D_r}| =$$

$$= \sum_{i=1}^{r} |Z_{D_i}| - \sum_{1 \le i < j \le r} |Z_{D_i} \cap Z_{D_j}| + \ldots + (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le r} |Z_{D_{i_1}} \cap Z_{D_{i_2}} \cap \ldots \cap Z_{D_{i_k}}| +$$

$$+ \ldots + (-1)^{r-1} |Z_{D_1} \cap Z_{D_2} \cap \ldots \cap Z_{D_r}|.$$

Let us consider the addends of the type $|Z_{D_{i_1}} \cap Z_{D_{i_2}} \cap \ldots \cap Z_{D_{i_k}}|$ where $k > m$. It is easy to see that each of them will contain at least 2 disjuncts D_{i_p} and D_{i_q} such that if D_{i_p} contain a variable x_j then D_{i_q} will contain its negation $\bar{x}_j$. In accordance with Lemma 1.1 $|Z_{D_{i_p}} \cap Z_{D_{i_q}}| = 0$ and therefore $|Z_{D_{i_1}} \cap Z_{D_{i_2}} \cap \ldots \cap Z_{D_{i_k}}| = 0$. So all the addends in formula (1) will be zeros after a fixed place to the end. This formula reduces to

(2) $$|Z_f| = |Z_{D_1 \wedge D_2 \wedge \ldots \wedge D_r}| = |Z_{D_1} \cup Z_{D_2} \cup \ldots \cup Z_{D_r}| =$$

$$= \sum_{i=1}^{r} |Z_{D_i}| - \sum_{1\le i<j\le r} |Z_{D_i} \cap Z_{D_j}| + \ldots + (-1)^{m-1} \sum_{1\le i_1<i_2<\ldots<i_m\le r} |Z_{D_{i_1}} \cap Z_{D_{i_2}} \cap \ldots \cap Z_{D_{i_m}}|.$$

After removing of too many addends the next question arises immediately: "Whether a formula (2) can be calculated in a polynomial time?". The answer is "No" (by now). Using the binomial bounds given in [3] and especially $\binom{n}{k} \ge (\frac{n}{k})^k$ we shall give one rough estimation only of the number of the additions and subtractions in formula (2). We set $r = 7m$ and so their number is:

$$(3) \qquad \binom{7m}{1} + \binom{7m}{2} + \ldots + \binom{7m}{m} \ge \left(\frac{7m}{1}\right)^1 + \left(\frac{7m}{2}\right)^2 + \ldots + \left(\frac{7m}{m}\right)^m >$$

$$> 7^m = (\sqrt[8]{7})^{8m} \approx (1,275)^{8m}.$$

3. Conclusions. It is important to note that we can describe algorithms which verify in polynomial time each necessary or sufficient condition which has been already formulated. Unfortunately, they refer to a small part of cases in the 3-Satisfiability problem. If we denote with N the number of all 3-CNForms of $\tilde{0}$ we shall obtain the next estimations:

$$2^{8m}(2^{-7} - 2^{-16}) \le N \le 2^{8m} - \left(2^{7m+1} - 2^{6m} + \sum_{i=0}^{7} \binom{8m}{i} - \sum_{i=0}^{7} \binom{6m}{i}\right).$$

The inequalities in formula (3) show that in the most of cases the 3-Satisfiability problem remains NP-hard.

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НЯКОИ НЕОБХОДИМИ И НЯКОИ ДОСТАТЪЧНИ УСЛОВИЯ В ЗАДАЧАТА ЗА 3-УДОВЛЕТВОРИМОСТ

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Така наречената задача за 3-удовлетворимост заема важно място в теорията за NP-пълнотата на задачи и алгоритми. Тази работа е продължение на друга предишна, в която се разглеждат и класифицират конюнктивни нормални форми на константата нула - тези, които съдържат до 3 букви на променливи (3-КНФорми). Тук ще изведем някои необходими и достатъчни условия, определящи дали дадени множества от 3-КНФорми са или не са форми на константата нула, които условия са полиномиално проверими. Дадени са и някои оценки за броя на тези форми.