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ALGORITHMS FOR SOLVING LINEAR MULTIPLICATIVE PROGRAMMING PROBLEMS

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A unified approach for minimizing and maximizing a product of two affine functions over a linear closed set is considered. No restrictions on the objective function and the constraint set are required. Finite algorithms are proposed and implemented via simplex method pivoting technique.

1. Introduction. We consider the linear multiplicative programming problem

LMP : minimize (maximize) $F(x) = P(x)Q(x)$ subject to $x \in \Omega$,

where $P(x)$ and $Q(x)$ are affine functions and $\Omega \in R^n$ is a closed linear set.

Recently, the interest in LMP increases, because of its applications in microelectronics, bond portfolio optimization, bicriterial optimization and so forth. All researchers focus their attention to the LMP for minimizing $F(x)$ (see, e.g., [1–7] and the references therein). The reason is, that usually it is supposed $P(x) \geq 0$ and $Q(x) \geq 0$ on Ω. Then the LMP for maximizing $F(x)$ could be solved by standard concave maximization techniques, while the LMP for minimizing $F(x)$ is a nonconvex global minimization problem. Different methods for finding a global minimum of LMP are proposed, many of them intended to solve more general problems, that could be applied to LMP (see, e.g., [1, 3, 5]). We specially note Konno and Kuno ([2,4]) among those that develop algorithms oriented directly to LMP, since they propose finite algorithms for finding a global minimum of LMP. However, even they assume that Ω is bounded.

In this paper, we consider a unified approach (Section 2) to solve LMP in the case of minimization and maximization, without any restrictions imposed on $F(x)$ and Ω. It is based on the research in [8]. The proposed algorithms (Section 3) are finite and can be implemented via parametric simplex method pivoting technique (Section 4).

2. Background. It is easy to see that LMP can be solved, if we can solve the following two subproblems

$\mathbf{LMP}^{++}_{\min}$: $\min \{F(x) = P(x)Q(x) \mid x \in \Omega_{++}\}$,

$\mathbf{LMP}^{++}_{\max}$: $\max \{F(x) = P(x)Q(x) \mid x \in \Omega_{++}\}$,

where $\Omega_{++}=\{x \in \Omega|\ P(x) \geq 0, Q(x) \geq 0\}$. Really, denote $\Omega_{+-}=\{x \in \Omega|\ P(x) \geq 0, Q(x) \leq 0\}$, $\Omega_{-+} = \{x \in \Omega|\ P(x) \leq 0, Q(x) \geq 0\}$ and $\Omega_{--} = \{x \in \Omega|\ P(x) \leq 0,$

$Q(x) \leq 0\big\}$. An optimal solution of LMP is among the optimal solutions of the subproblems for minimizing (maximizing) $F(x)$ over Ω_{++}, Ω_{+-}, Ω_{-+} and Ω_{--}. However, the optimization of $F(x)$ over the last three subsets can be obtained by solving one of the subproblems $\text{LMP}^{++}_{\min}$ and $\text{LMP}^{++}_{\max}$. For example, $\max\{P(x)Q(x) \mid x \in \Omega_{+-}\} = \min\{P(x)(-Q(x)) \mid P(x) \geq 0, -Q(x) \geq 0\}$.

The characteristics concerning $\text{LMP}^{++}_{\min}$ and $\text{LMP}^{++}_{\max}$ used here, are (see [8]):

C1. $F(x)$ is quasiconcave on Ω_{++}.

C2. $F(x)$ attains its minimum over Ω_{++} at a vertex of Ω_{++}. If $F(x)$ is bounded from above on Ω_{++}, it attains its maximum on an edge of Ω_{++}.

C3. $F(x)$ reaches its minimum (maximum) over Ω_{++} at x^*, iff there exists $t^* \in [0,1]$ such that $x^* =$ argmin(argmax) $\big\{L(x,t^*) = (1-t^*)P(x) + t^*Q(x) \mid\ x \in \Omega_{++}\big\}$.

The following sufficient condition also holds:

Theorem. *Let l be an edge of Ω_{++}, on which the affine function $L(x,t) = (1-t)P(x) + tQ(x)$ reaches its maximum for some value of $t \in [0,1]$. If $F(x) \not\equiv$ const on l, $x^* \in$ rint l and $x^* =$ argmax $\{F(x) \mid\ x \in l\ \}$, then $x^* =$ argmax $\{F(x) \mid\ x \in \Omega_{++}\}$.*

According to these characteristics, an optimal solution of $\text{LMP}^{++}_{\min}$ and $\text{LMP}^{++}_{\max}$ can be found by solving one-parametric linear problems:

$$\mathbb{LP}_{\min}: \qquad \min\big\{L(x,t) = (1-t)P(x) + tQ(x) \mid\ x \in\ \Omega_{++},\ t \in [0,1]\big\},$$

$$\mathbb{LP}_{\max}: \qquad \max\big\{L(x,t) = (1-t)P(x) + tQ(x) \mid\ x \in\ \Omega_{++},\ t \in [0,1]\big\},$$

respectively. This idea is briefly discussed in section 3 and implemented in section 4 via simplex-type technique. The algorithms in sections 3.1 and 3.2 are conventionally noted as *procedures* with arguments $P(x)$ and $Q(x)$, in order to facilitate the description of the general algorithms for solving LMP (Section 3.3).

3. Algorithms.

3.1. Procedure $\text{LMP}^{++}_{\min}\big(P(x), Q(x)\big)$. This procedure solves the problem $\text{LMP}^{++}_{\min}$. The idea is, as follows. $F(x)$ attains its minimum at a vertex $x^* \in \Omega_{++}$, for which there exists $t^* \in [0,1]$, such that $x^* = \text{argmin}\,\{L(x,t^*) = (1-t^*)P(x) + t^*Q(x) \mid x \in \Omega_{++}\}$ (see C2 and C3). We find t^* by solving $\text{LP}_{\min}$, increasing t from 0 to 1. x^* is among the found optimal vertices (basic optimal solutions) of $\text{LP}_{\min}$.

The algorithm starts with $t = 0$, i.e. with the linear program $\min\big\{L(x,0) = P(x) \mid x \in \Omega_{++}\big\}$. Let $\Omega_{++} \neq \emptyset$. $Q(x)$ will decrease at the next iterations and reach its minimum for $t = 1$: $\min\big\{L(x,1) = Q(x) \mid x \in \Omega_{++}\big\}$. Let $[t_0 = 0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_s, t_{s+1} = 1]$ be the sequence of intervals generated in the process of computations and $x^0, x^1, \cdots, x^s$ be the associated basic optimal solutions of $\text{LP}_{\min}$. Then $x^* = \text{argmin}\,\big\{F(x^k) \mid\ k = 0, 1, \cdots, s\ \big\}$ is a basic optimal solution of $\text{LMP}^{++}_{\min}$.

3.2. Procedure $\text{LMP}^{++}_{\max}\big(P(x), Q(x)\big)$. This procedure solves the problem $\text{LMP}^{++}_{\max}$. The idea is, as follows. If $F(x)$ is bounded from above on Ω_{++}, then it attains its maximum over Ω_{++} on an edge of Ω_{++} and every local maximum of $F(x)$ is global (see C1 and C2). If x^* is an optimal solution of $\text{LMP}^{++}_{\max}$, then there exists $t^* \in [0,1]$, such that $x^* = \text{argmax}\,\{L(x,t^*) = (1-t^*)P(x) + t^*Q(x) \mid x \in \Omega_{++}\}$ (see C3). We find t^* by solving $LP_{\max}$, increasing $t \in [0,1]$.

The algorithm starts with $t = 0$, i.e. with the linear program $\max\{L(x,0) = P(x)|\ Ax = b,\ x \geq \mathbf{0}\}$. Let $\Omega_{++} \neq \emptyset$. $Q(x)$ will increase at the next iterations and reach its maximum (if $F(x)$ is bounded from above on Ω_{++}) when $t = 1$: $\max\ \{L(x,1) = Q(x)| x \in \Omega_{++}\}$. We perform the iterations just while $F(x)$ also increases, i.e. untill the edge containing x^* is reached ($t = t^*$). Consider the kth iteration ($k \geq 0$) in the general case, when $F(x)$ is not an affine function, i.e. $P(x) \not\equiv$const or/and $Q(x) \not\equiv$const on Ω_{++}. Let x^k be the optimal vertex of $\mathrm{LP}_{\max}$, found for $t = t_k$ and optimal for all $t \in [t_k, t_{k+1}] \subset [0,1]$. Denote by J_N^k the index set corresponding to the nonbasic variables of x^k and by d^j ($j \in J_N^k$) a direction of the edge $l_j \subset \Omega_{++}$, which emanates from x^k and along which the nonbasic variable x_j increases. The possible cases are:

– $L(x, t_k) \to +\infty$. Then $F(x)$ is also unbounded from above on Ω_{++};

– $\nabla^T F(x^k) d^j \leq 0$ for every $j \in J_N^k$, i.e. $F(x)$ decreases along all edges, which emanate from x^k and belong to Ω_{++}. Then x^k is a basic optimal solution of $\mathrm{LMP}_{\max}^{++}$ (see C1);

– There exists $r \in J_N^k$ such that $\nabla^T F(x^k) d^r > 0$. Denote by l_r the edge connecting x^k with the next vertex $x^{k+1} = x^k + \theta d_r$ ($\theta > 0$), optimal for $t = t_{k+1}$. $F(x)$ increases along the edge l_r. We check whether an optimal solution of $\mathrm{LMP}_{\max}^{++}$ lies on l_r, using a line search. Let $\mu^* = \mathrm{argmax}\ \{\varphi(\mu) = F(x^k + \mu d_r)|\ \mu \geq 0\ \}$. If $\mu^* < \theta$, then $x^* = x^k + \mu^* d_r \in \mathrm{rint}\ l_r$ is an optimal solution of $\mathrm{LMP}_{\max}^{++}$ (see the Theorem). Note that, because of the special kind of $F(x)$, μ^* is computed through a simple explicit formula (see Section 4);

– None of these cases occurs. We go on with the next iteration.

3.3. Solving LMP. If we know that $F(x)$ has a constant sign on Ω, then the LMP can be solved only by Procedure $\mathrm{LMP}_{\max}^{++}$ or Procedure $\mathrm{LMP}_{\min}^{++}$. In the general case, the algorithms given below take into account:

$$\begin{aligned} &\max_{x\in\Omega} F(x) = \max\{\max_{x\in\Omega_{++}} F(x),\ \max_{x\in\Omega_{--}} F(x)\} = \alpha\ (\alpha < +\infty \text{ or } \alpha = +\infty), \\ &\min_{x\in\Omega} F(x) = \min\{\min_{x\in\Omega_{+-}} F(x),\ \min_{x\in\Omega_{-+}} F(x)\} = \beta\ (\beta > -\infty \text{ or } \beta = -\infty), \end{aligned} \tag{3.1}$$

if $\Omega_{++} \cup \Omega_{--} \neq \emptyset$ and $\Omega_{+-} \cup \Omega_{-+} \neq \emptyset$, respectively. Otherwise,

$$\begin{aligned} &\text{if } \Omega_{++} \cup \Omega_{--} = \emptyset,\ \text{then } \Omega_{+-} = \emptyset \text{ or/and } \Omega_{-+} = \emptyset, \\ &\text{if } \Omega_{+-} \cup \Omega_{-+} = \emptyset,\ \text{then } \Omega_{++} = \emptyset \text{ or/and } \Omega_{--} = \emptyset. \end{aligned} \tag{3.2}$$

Algorithm for solving LMP: min $\{F(x) = P(x)Q(x)|\ x \in \Omega\}$.

I. Solve the problems

$\min\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{+-}\}$ by procedure $\mathrm{LMP}_{\max}^{++}\ (P(x), -Q(x))$,

$\min\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{-+}\}$ by procedure $\mathrm{LMP}_{\max}^{++}\ (-P(x), Q(x))$.

If $\Omega_{+-} \cup \Omega_{-+} = \emptyset$, go to II. Otherwise, LMP is solved and $\min\ \{F(x)|\ x \in \Omega\} = \beta$ (see (3.1)). If $\beta = -\infty$, then $F(x) \to -\infty$ on Ω.

II. Solve the problems

$\min\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{++}\}$ by procedure $\mathrm{LMP}_{\min}^{++}\ (P(x), Q(x))$,

$\min\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{--}\}$ by procedure $\mathrm{LMP}_{\min}^{++}\ (-P(x), -Q(x))$.

If $\Omega_{++} \cup \Omega_{--} = \emptyset$, then $\Omega = \emptyset$. Otherwise (see (3.2)),

$$\min_{x\in\Omega} F(x) = \min_{x\in\Omega_{++}} F(x) > -\infty \quad \text{or} \quad \min_{x\in\Omega} F(x) = \min_{x\in\Omega_{--}} F(x) > -\infty.$$

Algorithm for solving LMP: max $\{F(x) = P(x)Q(x)|\ x \in \Omega\}$

I. Solve the problems

$$\max\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{++}\} \quad \text{by procedure } \mathrm{LMP}^{++}_{\max}\ \big(P(x), Q(x)\big),$$
$$\max\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{--}\} \quad \text{by procedure } \mathrm{LMP}^{++}_{\max}\ \big(-P(x), -Q(x)\big).$$

If $\Omega_{++} \cup \Omega_{--} = \emptyset$, go to II. Otherwise, LMP is solved and $\max\{F(x)|\ x \in \Omega\} = \alpha$ (see (3.1)). If $\alpha = +\infty$, then $F(x) \to +\infty$ on Ω.

II. Solve the problems

$$\max\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{+-}\} \quad \text{by procedure } \mathrm{LMP}^{++}_{\min}\ \big(P(x), -Q(x)\big),$$
$$\max\ \{F(x) = P(x)Q(x)|\ x \in \Omega_{-+}\} \quad \text{by procedure } \mathrm{LMP}^{++}_{\min}\ \big(-P(x), Q(x)\big).$$

If $\Omega_{+-} \cup \Omega_{-+} = \emptyset$, then $\Omega = \emptyset$. Otherwise (see (3.2)),

$$\max_{x\in\Omega} F(x) = \max_{x\in\Omega_{+-}} F(x) < +\infty \quad \text{or} \quad \max_{x\in\Omega} F(x) = \max_{x\in\Omega_{-+}} F(x) < +\infty.$$

4. Implementation. Here we apply a simplex-type technique to the algorithms in sections 3.1 and 3.2 and deduce some formulas connected with them. For this purpose we specify $P(x) = p^T x + p_0$ and $Q(x) = q^T x + q_0$, and suppose that Ω_{++} is given in the standard form $\Omega_{++} = \big\{x \in R^n|\ \ Ax = b,\ \ x \geq \mathbf{0}\big\}$, where $p = (p_1, p_2, \cdots, p_n)^T$, $q = (q_1, q_2, \cdots, q_n)^T$, $p_0, q_0 \in R^1$, $b \in R^m$, $b \geq \mathbf{0}$, $A \in R^{m\times n}$, $m < n$ (by $\mathbf{0}$ is denoted a zero vector).

To solve $\mathrm{LP}_{\min}$ or $\mathrm{LP}_{\max}$ means to find the intervals $[t_0 = 0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_s, t_{s+1} = 1]$ and the corresponding basic optimal solutions x^0, x^1, $\cdots$, x^s, where x^k $(k = 0, 1, \cdots, s)$ is optimal for $t \in [t_k, t_{k+1}]$. Consider the kth iteration. Let x^k be the current basic optimal solution found for $t = t_k$. Denote by B and N the corresponding basic and nonbasic matrix, respectively. Let x_B, p_B and q_B be the basic part of the vector x, p and q, respectively, and x_N, p_N and q_N be the nonbasic part of x, p, q, respectively. Set $I = \{1, 2, \cdots, m\}$, $J = \{1, 2, \cdots, n\}$ and denote by $J^k_B = \{s_1, s_2, \cdots, s_m\}$ and $J^k_N = J \setminus J^k_B$ the index set corresponding to the basic and nonbasic variables of x^k, respectively. Let $x_B + \alpha^k_N x_N = \beta^k$ be the presentation of the system $Ax = b$ regarding B, where $\alpha^k_N = B^{-1}N = (\alpha^k_{ij})_{m\times(n-m)}$ $(i \in I, j \in J^k_N)$, $\beta^k = B^{-1}b = (\beta^k_1, \cdots, \beta^k_m)^T$. We have $x^k_N = \mathbf{0}$, $x^k_B = \beta^k$ and

$$P(x) = p_0 + p^T_B \beta^k + (p^T_N - p^T_B \alpha^k_N) x_N \quad = \delta^k_0 + \delta^k_N x_N,$$
$$Q(x) = q_0 + q^T_B \beta^k + (q^T_N - q^T_B \alpha^k_N) x_N \quad = \Delta^k_0 + \Delta^k_N x_N,$$

$L(x, t) = (1-t)P(x) + tQ(x) = (1-t)\delta^k_0 + t\Delta^k_0 + \big[(1-t)\delta^k_N + t\Delta^k_N\big] x_N = \sigma^k_0(t) + \sigma^k_N(t) x_N$, where

$$\begin{array}{lllll} \delta^k_0 & = p_0 + p^T_B \beta^k & = P(x^k), & \delta^k_N & = p^T_N - p^T_B \alpha^k_N, \\ \Delta^k_0 & = q_0 + q^T_B \beta^k & = Q(x^k) & \Delta^k_N & = q^T_N - q^T_B \alpha^k_N, \\ \sigma^k_0(t) & = (1-t)\delta^k_0 + t\Delta^k_0 & = L(x^k, t), & \sigma^k_N(t) & = (1-t)\delta^k_N + t\Delta^k_N. \end{array} \quad (4.1)$$

The basis of x^k is optimal for all $t \in [t_k, t_{k+1}] \subset [0,1]$ that satisfy the system of inequalities

$$(4.2) \qquad \begin{array}{ll} \sigma_j^k(t) = (1-t)\delta_N^k + t\Delta_N^k \geq 0, \quad j \in J_N^k, & \text{in the case of } \mathrm{LP}_{\min}, \\ \sigma_j^k(t) = (1-t)\delta_N^k + t\Delta_N^k \leq 0, \quad j \in J_N^k, & \text{in the case of } \mathrm{LP}_{\max}. \end{array}$$

Let $l_j \in \Omega_{++}$ ($j \in J_N^k$) be an edge that emanates from x^k and along which the nonbasic variable x_j increases. Let $d^j = (d_1^j, \cdots, d_n^j)$ be a direction of l_j with components: $d_j^j = 1$, $d_{s_i}^j = -\alpha_{ij}^k$ for $s_i \in J_B^k$, $i \in I$ and $d_h^j = 0$ for $h \in J_N^k \setminus \{j\}$. Because of (4.1), we have

$$(4.3) \quad p^T d^j = p_j - \sum_{i=1}^m p_{s_i}\alpha_{ij}^k = \delta_j^k \quad \text{and} \quad q^T d^j = q_j - \sum_{i=1}^m q_{s_i}\alpha_{ij}^k = \Delta_j^k \quad \text{for } j \in J_N^k,$$

$$g_j^k = \nabla^T F(x^k) d^j = (\Delta_0^k p^T + \delta_0^k q^T) d^j = \Delta_0^k \delta_j^k + \delta_0^k \Delta_j^k > 0.$$

We pass to the next vertex x^{k+1} choosing an edge l_j, $j \in J_N^k$, as follows:

In the case of $\mathrm{LP}_{\min}$, $Q(x)$ has to decrease along l_j, i.e. $j \in J_-^k = \{j \in J_N^k \mid \Delta_j^k < 0\}$. If $J_-^k = \emptyset$, then $t_{k+1} = 1$, i.e. $\mathrm{LP}_{\min}$ is solved and an optimal solution of $\mathrm{LMP}_{\min}^{++}$ is among the found basic optimal solutions of $\mathrm{LP}_{\min}$.

In the case of $\mathrm{LP}_{\max}$, $F(x)$ has to increase along l_j, i.e $j \in J_+^k = \{j \in J_N^k \mid g_j^k > 0\}$. If $\alpha_{ij}^k \leq 0$ for $i \in I$ and $j \in J_+^k \neq \emptyset$, then $F(x) \to +\infty$ on Ω_{++}. If $J_+^k = \emptyset$, then x^k is a basic optimal solution of $\mathrm{LMP}_{\max}^{++}$.

If none of these cases occurs, we determine the maximal solution t^{k+1} of (4.2) through

$$(4.4) \qquad \begin{array}{ll} t_{k+1} = \min\Big\{ t_j^k = \dfrac{\delta_j^k}{\delta_j^k - \Delta_j^k} \mid j \in J_-^k \neq \emptyset \Big\} = t_r^k & \text{in the case of } \mathrm{LP}_{\min}, \\ t_{k+1} = \min\Big\{ t_j^k = \dfrac{\delta_j^k}{\delta_j^k - \Delta_j^k} \mid j \in J_+^k \neq \emptyset \Big\} = t_r^k & \text{in the case of } \mathrm{LP}_{\max} \end{array}$$

and obtain that l_r is the edge leading to the next vertex x^{k+1}, optimal for $t = t_{k+1}$: $x^{k+1} = x^k + \theta d^r$, where

$$(4.5) \qquad \theta = \min\Big\{ \frac{\beta_i^k}{\alpha_{ir}^k} \mid \alpha_{ir}^k > 0, \ i \in I \Big\} = \frac{\beta_l^k}{\alpha_{lr}^k}.$$

The known recurrent formulas connected with the basis of x^{k+1} are:

$$(4.6) \qquad \begin{array}{l} \beta_l^{k+1} = \theta, \quad \beta_i^{k+1} = \beta_i^k - \theta\alpha_{ir}^k, \quad i \in I, \ i \neq l, \\ \alpha_{lj}^{k+1} = \dfrac{\alpha_{lj}^k}{\alpha_{lr}^k}, \alpha_{ij}^{k+1} = \alpha_{ij}^k - \dfrac{\alpha_{lj}^k}{\alpha_{lr}^k}\alpha_{ir}^k, i \in I, \ i \neq l, \ j \in J_N^{k+1} \\ \delta_0^{k+1} = \delta_0^k + \theta\delta_r^k = P(x^{k+1}) \quad \delta_j^{k+1} = \delta_j^k - \dfrac{\alpha_{lj}^k}{\alpha_{lr}^k}\delta_r^k, \quad j \in J_N^{k+1}, \\ \Delta_0^{k+1} = \Delta_0^k + \theta\Delta_r^k = Q(x^{k+1}) \quad \Delta_j^{k+1} = \Delta_j^k - \dfrac{\alpha_{lj}^k}{\alpha_{lr}^k}\Delta_r^k, \quad j \in J_N^{k+1}. \\ x_r^{k+1} = \beta_l^{k+1}, \quad x_{s_i}^{k+1} = \beta_i^{k+1} \quad \text{for} \quad s_i \in J_B^{k+1}, i \in I, i \neq l, \\ x_j^{k+1} = 0 \quad \text{for} \quad j \in J_N^{k+1}. \end{array}$$

In the case of $\mathrm{LP}_{\max}$, before passing to x^{k+1}, we perform the line search max $\{\varphi(\mu) = F(x^k + \mu d^r) \mid \mu \geq 0\}$ in order to check whether $F(x)$ attains its maximum over l_r at an interior point of l_r. Taking into account (4.1) and (4.3) for $j = r$, we obtain

$$\begin{aligned}\varphi(\mu) &= F(x^k + \mu d^r) = [p^T(x^k + \mu d^r) + p_0][q^T(x^k + \mu d^r) + q_0] \\ &= (p^T x^k + p_0 + \mu p^T d^r)(q^T x^k + q_0 + \mu q^T d^r) \\ &= (\delta_0^k + \mu\delta_r^k)(\Delta_0^k + \mu\Delta_r^k) = \delta_0^k\Delta_0^k + \mu(\Delta_0^k\delta_r^k + \delta_0^k\Delta_r^k) + \mu^2\delta_r^k\Delta_r^k\end{aligned}$$

and $\dfrac{d\varphi(\mu)}{d\mu} = \Delta_0^k\delta_r^k + \delta_0^k\Delta_r^k + 2\mu\delta_r^k\Delta_r^k = g_r^k + 2\mu\delta_r^k\Delta_r^k = 0$, from where the solution is

$$\mu^* = \frac{-g_r^k}{2\delta_r^k\Delta_r^k}. \tag{4.7}$$

If $\mu^* < \theta$, then $x^* = x^k + \mu^* d^r \in \text{rint } l_r$ and x^* is an nonbasic optimal solution of the problem $\text{LMP}_{\max}^{++}$. Otherwise, we pass to the vertex x^{k+1} and go on solving $\text{LP}_{\max}$.

4.1. Algorithm for solving $\text{LMP}_{\min}^{++}$:

0. Set $k = 0$, $t_0 = 0$ and solve the linear program $\text{LP}_{\min}$ for $t = t_0$. If $\Omega_{++} = \emptyset$, go to 5. Otherwise, let x^0 be the found basic optimal solution . The elements α_{ij}^0, β_i^0, δ_0^0 and δ_j^0 of δ_N^0 $(i \in I,\ j \in J_N^0)$ connected with x^0 are already computed. Calculate now Δ_0^0 and Δ_j^0 for $j \in J_N^0$ through (4.1) $(k = 0)$ and set $\Delta_j^0 = 0$ for $j \in J_B^0$. Check:

– If $\delta_j^0 = 0$ for $j \in J_N^0$, then $P(x) \equiv \delta_0^0 = P(x^0)$ on Ω_{++}. If $\delta_0^0 = 0$ then $F(x) \equiv 0$ on Ω_{++} and x^0 is an optimal solution of $\text{LMP}_{\min}^{++}$ – go to 5. Otherwise, solve the linear program $\min\{F(x) = \delta_0^0 Q(x)|\ x \in \Omega_{++}\}$ and go to 5;

– If $\Delta_j^0 = 0$ for $j \in J_N^0$, then $Q(x) \equiv \Delta_0^0 = Q(x^0)$ on Ω_{++}. If $\Delta_0^0 = 0$ then $F(x) \equiv 0$ on Ω_{++} and x^0 is an optimal solution of $\text{LMP}_{\min}^{++}$ – go to 5. Otherwise, solve the linear program $\min\{F(x) = \Delta_0^0 P(x)|\ x \in \Omega_{++}\}$ and go to 5;

– Set $x^* = x^0$ and $F^* = F(x^0) = \delta_0^0\Delta_0^0$.

1. Form the set $J_-^k = \{j \in J_N^k \mid \Delta_j^k < 0\}$. If $J_-^k = \emptyset$, go to 4.

2. Pass to x^{k+1}:

– Calculate t_j^k for $j \in J_-^k$ and determine t_{k+1} according to (4.4). Calculate θ through (4.5). Set $J_B^{k+1} = \{x_r\} \cup J_B^k \setminus \{x_{s_l}\}$ and $J_N^{k+1} = J \setminus J_B^{k+1}$.

– Calculate α_{ij}^{k+1} and β_i^{k+1} $(i \in I,\ j \in J_N^{k+1})$, δ_0^{k+1}, Δ_0^{k+1}, δ_j^{k+1} and Δ_j^{k+1} $(j \in J_N^{k+1})$ through (4.6). Set $\delta_j^{k+1} = 0$ and $\Delta_j^{k+1} = 0$ for $j \in J_B^{k+1}$.

3. If $F(x^{k+1}) = \delta_0^{k+1}\Delta_0^{k+1} < F^*$, set $F^* = F(x^{k+1})$ and $x^* = x^{k+1}$ (the components of x^{k+1} are given by (4.6)). Set $k := k + 1$ and go to 1.

4. x^* is a basic optimal solution of $\text{LMP}_{\min}^{++}$ and $F(x^*) = F^*$.

5. End.

4.2. Algorithm for solving $\mathrm{LMP}^{++}_{\max}$:

0. Set $k = 0$, $t_0 = 0$ and solve the linear program $\mathrm{LP}_{\max}$ for $t = t_0$. If $\Omega_{++} = \emptyset$, then go to 6. Otherwise, let x^0 is the last found basic feasible solution. The elements α^0_{ij}, β^0_i, δ^0_0 and δ^0_j of δ^0_N ($i \in I$, $j \in J^0_N$) connected with x^0 are already computed. Calculate now Δ^0_0 and Δ^0_j for $j \in J^0_N$ by using (4.1) ($k = 0$) and set $\Delta^0_j = 0$ for $j \in J^0_B$. Check consecutively:

– If $\delta^0_j = 0$ for $j \in J^0_N$, then $P(x) \equiv \delta^0_0 = P(x^0)$ on Ω_{++}. If $\delta^0_0 = 0$ then $F(x) \equiv 0$ on Ω_{++} and x^0 is a basic optimal solution of $\mathrm{LMP}^{++}_{\min}$ – go to 6. Otherwise, solve the linear program $\min\,\{F(x) = \delta^0_0 Q(x)|\ x \in \Omega_{++}\}$ and go to 6.

– If $\Delta^0_j = 0$ for $j \in J^0_N$, then $Q(x) \equiv \Delta^0_0 = Q(x^0)$ on Ω_{++}. If $\Delta^0_0 = 0$ then $F(x) \equiv 0$ on Ω_{++} and x^0 is a basic optimal solution of $\mathrm{LMP}^{++}_{\min}$ – go to 6. Otherwise, solve the linear program $\min\,\{F(x) = \Delta^0_0 P(x)|\ x \in \Omega_{++}\}$ and go to 6.

– There exists $\delta^0_j > 0$ ($j \in J^0_N$) and $\alpha^0_{ij} \le 0$ for $i \in I$, then $P(x) \to +\infty$ on Ω_{++}. Hence $F(x) \to +\infty$ on Ω_{++}. Go to 6.

If none of the cases above occurs, then x^0 is a basic optimal solution of $\mathrm{LP}_{\max}$.

1. If there exists $\Delta^k_j > 0$ ($j \in J^k_N$) and $\alpha^k_{ij} \le 0$ for $i \in I$, then $Q(x) \to +\infty$ on Ω_{++}. Hence, $F(x) \to +\infty$ on Ω_{++} – go to 6.

2. Calculate $g^k_j = \Delta^k_0 \delta^k_j + \delta^k_0 \Delta^k_j$ for $j \in J^k_N$ and form the set $J^k_+ = \{j \in J^k_N \mid g^k_j > 0\}$. If $J^k_+ = \emptyset$, then x^k is a basic optimal solution of $\mathrm{LMP}^{++}_{\max}$ and $F(x^k) = \delta^k_0 \Delta^k_0$ (the components of x^{k+1} are given by (4.6)). Go to **6**.

3 Calculate t^k_j for $j \in J^k_+$ and determine t_{k+1} according to (4.4). Calculate θ through (4.5).

4. Check for nonbasic optimal solution, as follows. Compute μ^* through (4.7). If $\mu^* \ge \theta$, go to 5. Otherwise $x^* = x^k + \mu^* d^r$ is an optimal solution of $\mathrm{LMP}^{++}_{\max}$ with components

$$x^*_{s_i} = \beta^k_i - \mu^* \alpha^k_{ir}\ (s_i \in J^k_B, i \in I), \quad x^*_r = \mu^*, \quad x^*_j = 0\ (j \in J^k_N \setminus \{r\}).$$

Go to **6**.

5. Pass to x^{k+1}, as follows. Set $J^{k+1}_B = \{x_r\} \cup J^k_B \setminus \{x_{s_l}\}$ and $J^{k+1}_N = J \setminus J^{k+1}_B$. Calculate α^{k+1}_{ij}, β^{k+1}_i ($i \in I$, $j \in J^{k+1}_N$), δ^{k+1}_0, Δ^{k+1}_0, δ^{k+1}_j and Δ^{k+1}_j ($j \in J^{k+1}_N$) through (4.6). Set $\delta^{k+1}_j = 0$ and $\Delta^{k+1}_j = 0$ for $j \in J^{k+1}_B$. Set $k := k + 1$ and go to 1.

6. End.

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АЛГОРИТМИ ЗА РЕШАВАНЕ НА ЗАДАЧИ НА ЛИНЕЙНОТО МУЛТИПЛИКАТИВНО ОПТИМИРАНЕ

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Разглежда се единен подход за минимизиране и максимизиране на произведение от две афинни функции върху затворена линейна област, без ограничения върху целевата функция и областта. Предложени са крайни алгоритми за решаване на поставените задачи, които се свеждат до решаване на задачи на линейното оптимиране с параметър в целевата фуркция.