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CUBIC INTERSECTIONS BY MOVING PLANE

Albena Jordanova, Victoria Daneva

We investigate cubic intersections by moving plane, which depends on three parameters. For this we prove the main theorem and find the area function (Theorem 2). We show that the area function is a smooth function and differentiable once only. In many special cases this function has at least two points of maximum and four inflection points. For this purpose we also use computer-based methods.
We recommend some special cases that can be used in school mathematics. In this way one can integrate together plane geometry, solid geometry, algebra, analysis and computer science.

We consider a cube$ABCDA_1B_1C_1D_1$ with edge a, and a plane ε_{MNK} determined by:

(1) $$\overrightarrow{BM} = m\overrightarrow{BA},\ \overrightarrow{BN} = n\overrightarrow{BC},\ \overrightarrow{BK} = k\overrightarrow{BB_1}:\ m, n, k \in (-\infty, +\infty).$$

This plane cuts the lines of the other edges at the points: $P, S, T, U, V, W, X, Y, Z$, given by the equations:

(2) $$\begin{aligned} &\overrightarrow{AT} = n\frac{m-1}{m}\overrightarrow{AD}, \quad \overrightarrow{CV} = m\frac{n-1}{n}\overrightarrow{CD}, \quad \overrightarrow{DY} = k\frac{mn-m-n}{mn}\overrightarrow{DD_1},\\ &\overrightarrow{AS} = k\frac{m-1}{m}\overrightarrow{AA_1}, \quad \overrightarrow{B_1U} = m\frac{k-1}{k}\overrightarrow{B_1A_1}, \quad \overrightarrow{A_1Z} = n\frac{km-k-m}{km}\overrightarrow{A_1D_1},\\ &\overrightarrow{CW} = k\frac{n-1}{n}\overrightarrow{CC_1}, \quad \overrightarrow{B_1X} = n\frac{k-1}{k}\overrightarrow{B_1C_1}, \quad \overrightarrow{C_1P} = m\frac{kn-k-n}{kn}\overrightarrow{C_1D_1}. \end{aligned}$$

We shall give some short proofs of these relations:

The equality $\overrightarrow{BM} = m\overrightarrow{BA}$ is equivalent to the relation $\overline{BM} = m\overline{BA}$, where $\overline{BM}$ and $\overline{BA}$ are the algebraic measures respectively of $\overrightarrow{BM}$ and $\overrightarrow{BA}$.

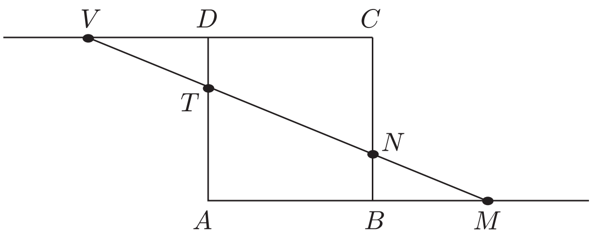


Fig. 1

The triangles BMN and CVN are similar (Fig. 1). It follows:

$$\frac{\overline{BN}}{\overline{CN}} = \frac{\overline{BM}}{\overline{CV}} \iff \overline{CV} = \frac{\overline{BM}}{\overline{BN}}\overline{CN};$$

Since $\overline{CN} = \overline{CB} + \overline{BN}$, it follows $\overline{CV} = m\frac{n-1}{n}\overline{CD}$, which means, that $\overrightarrow{CV} = m\frac{n-1}{n}\overrightarrow{CD}$.

In the same way one can prove all relations (2).

In this paper we consider the case $m < 0$, $0 < n < 1$, k – arbitrary. There are two possible cases:

Case 1: The point T belongs to the edge AD (in such a case the point V does not belong to the edge DC).

Case 2: The point V belongs to the edge CD (in such a case the point T does not belong to the edge AD) (fig. 2).

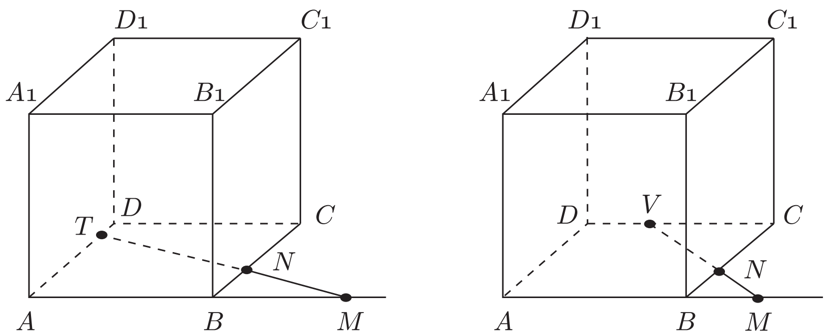


Fig. 2

Now we shall consider the first case, i.e. the point T is from the intersection of the cube and the plane ε. Then from the representation: $\overrightarrow{AT} = n\dfrac{m-1}{m}\overrightarrow{AD}$, it follows $0 < n\dfrac{m-1}{m} < 1$ which is equivalent to the inequality: $mn - n - m > 0$.

Lemma 1. *The point T belongs to the section $\iff$ $mn - n - m > 0$.*

The relation: $mn = n = m = 0$ determines a hyperbole with vertices $(0, 0)$ and $(2, 2)$ and asymptotes $m = 1$ and $n = 1$ (Fig. 3).

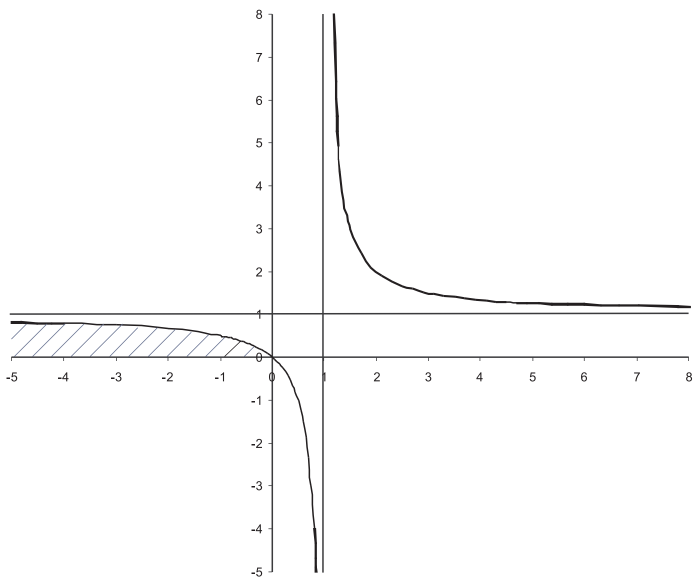


Fig. 3

One can easily see that we have in mind all points $X(m,n)$ for which $m<0$, $0<n<1$ and which are placed under the hyperbole (Fig. 3).

Let us designate $k_0=\dfrac{mn}{mn-m-n}$.

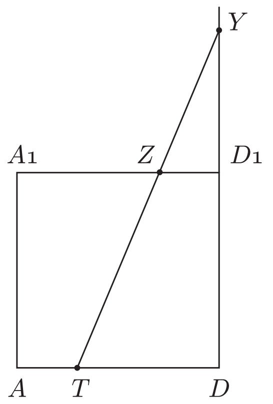


Fig. 4

Lemma 2. *The following inequalities are valid:*

$$k_0<\frac{n}{n-1}<0<\frac{m}{m-1}<1.$$

The proofs follow directly.

A) Let $k<k_0$.

From the equations (2) it follows immediately that the point Y belongs to the ray ${}^{\leftarrow}D_1$D (where the ray originating at the point D_1 and does not containing a point D is designated ${}^{\leftarrow}D_1D$) then the point Z is from the edge A_1D_1 (Fig. 4).

B) Let $k<\dfrac{n}{n-1}$. From (2) $\Rightarrow W\in {}^{\leftarrow}C_1C \iff X\in B_1C_1$.

From A) and B) it follows:

Lemma 3. *If $k\le k_0$ the section is quadrilateral $NXZT$.*

C) Let $k_0<k<\dfrac{n}{n-1}$.

Then Y belongs to the edge DD_1, consequently $Z\in {}^{\leftarrow}D_1A_1$ and $P\in D_1C_1$.

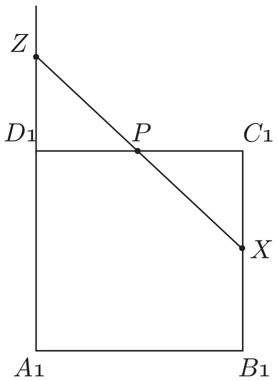


Fig. 5

Lemma 4. *If $k_0<k<\dfrac{n}{n-1}$ the intersection is a pentagon $NXPYT$.*

D) If $\dfrac{n}{n-1}\le k<0$ then $W\in CC_1$ and $Y\in DD_1$.

Lemma 5. *If $\dfrac{n}{n-1}\le k<0$, the section is quadrilateral $NWYT$.*

E) If $0<k\le\dfrac{m}{m-1}$, then from (2) it follows that S belongs to the edge AA_1.

Lemma 6. *If $0<k\le\dfrac{m}{m-1}$, the intersection is quadrilateral $NKST$.*

F) If $k>\dfrac{m}{m-1}$, then $S\in {}^{\leftarrow}A_1A$.

G) If $\dfrac{m}{m-1}<k<1$. From (2) and F) it follows: $U\in A_1B_1$, $Z\in A_1D_1$.

Lemma 7. *If $\dfrac{m}{m-1}<k<1$, the intersection is a pentagon $NKUZT$.*

And finally let us $k\ge 1$.

H) If $k\ge 1$ then $X\in B_1C_1$ and $S\in {}^{\leftarrow}A_1A$.

Lemma 8. *If $k\ge 1$, the section is quadrilateral $NXZT$.*

Theorem 1(the main section theorem). *For $k \neq 0$ the figure of the section depends on k and:*

a) If $k \in (-\infty, k_0]$, the intersection is a quadrilateral $NXZT$;

b) If $k \in \left(k_0, \dfrac{n}{n-1}\right)$, the intersection is a pentagon $NXPYT$;

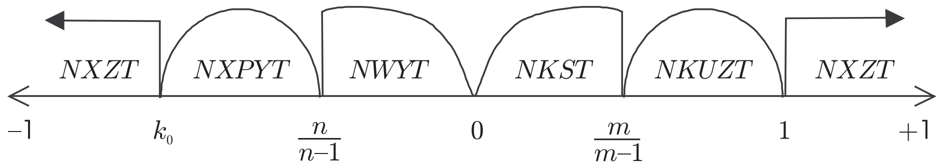
c) If $k \in \left[\dfrac{n}{n-1}, 0\right)$, the intersection is a quadrilateral $NWYT$;

d) If $k \in \left(0, \dfrac{m}{m-1}\right]$, the intersection is a quadrilateral $NKST$;

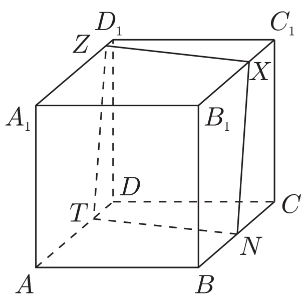
e) If $k \in \left(\dfrac{m}{m-1}, 1\right)$, the intersection is a pentagon $NKUZT$;

f) If $k \in [1, +\infty)$, the section is a quadrilateral $NXZT$;

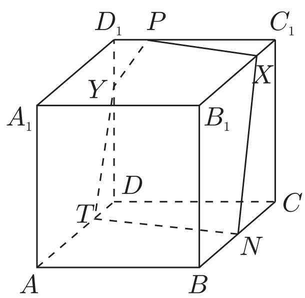
Or shortly:



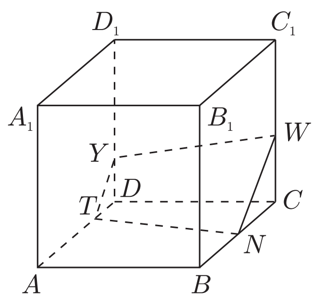
The following figures illustrate this theorem in some special cases: $m = -2$; $n = \dfrac{1}{2}$ then $k_0 = -2$, $\dfrac{n}{n-1} = -1$, $\dfrac{m}{m-1} = \dfrac{2}{3}$:



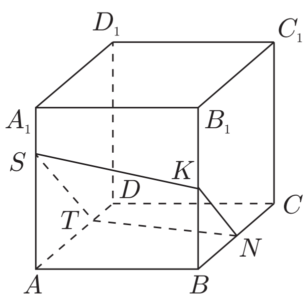
k=–3



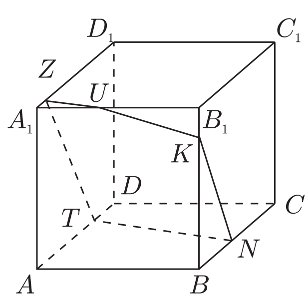
k=–3/2



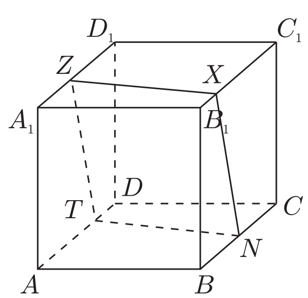
k=–1/2



k=1/2



k=3/4



k=2

Theorem 2. *For the area function $\sigma(k)$ holds:*

$$(3)\quad \begin{aligned}
&\sigma_1(k)=\sigma_{NXZT}=\frac{a^2}{km}\sqrt{k^2m^2+m^2n^2+n^2k^2}, \quad k\in(-\infty,k_0],\\
&\sigma_2(k)=\sigma_{NXPYT}=\frac{a^2}{2}\left(\frac{2}{km}-\left(\frac{kmn-km-kn-mn}{kmn}\right)^2\right)\sqrt{m^2n^2+m^2k^2+n^2k^2},\\
&\qquad\qquad\qquad\qquad\qquad\qquad k\in\left(k_0,\frac{n}{n-1}\right),\\
&\sigma_3(k)=\sigma_{NWYT}=\frac{a^2}{2m^2n}(2mn-2n-m)\sqrt{m^2n^2+m^2k^2+n^2k^2},\ k\in\left[\frac{n}{n-1},0\right),\\
&\sigma_4(k)=\sigma_{NKST}=\frac{a^2}{2m^2}(1-2m)\sqrt{m^2n^2+m^2k^2+n^2k^2},\ k\in\left(0,\frac{m}{m-1}\right],\\
&\sigma_5(k)=\sigma_{NKUZT}=-\frac{a^2}{2m}\left(\frac{2}{k}+m\left(\frac{k-1}{k}\right)^2\right)\sqrt{k^2m^2+m^2n^2+n^2k^2},\\
&\qquad\qquad\qquad\qquad\qquad\qquad k\in\left(\frac{m}{m-1},1\right),\\
&\sigma_6(k)=\sigma_{NXZT}=-\frac{a^2}{km}\sqrt{k^2m^2+m^2n^2+n^2k^2},\ k\in[1,+\infty).
\end{aligned}$$

Proof. We use the formula $\sigma^1=\sigma\cos\varphi$, where σ is the area of a plane polygon, σ^1 – the area of its orthogonal projection, and φ is the angle between the planes of the polygon and its projection.

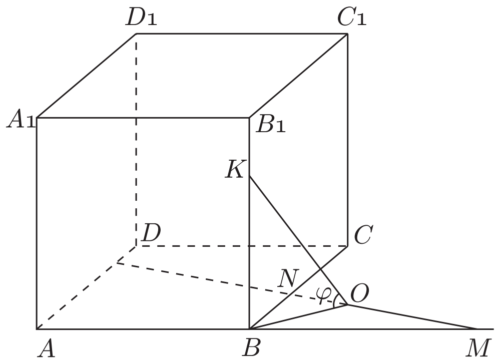


Fig. 6

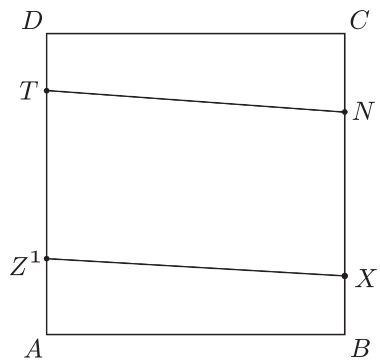


Fig. 7

Let us consider the plane trough BK and orthogonal to MN. From $\triangle MBN$ we have $\overline{BO}.\overline{MN}=ma.na$, where $O\in MN$ and $BO\perp MN$, $\varphi=\sphericalangle BOK$. (Fig. 6)
Since $\overline{MN}=a\sqrt{m^2+n^2}$, it follows $\overline{BO}=-\dfrac{mna}{\sqrt{m^2+n^2}}$ and $\cos\varphi=-\dfrac{mn}{\sqrt{m^2n^2+k^2m^2+k^2n^2}}$.
To find the area σ_6, let NX^1Z^1T be the projection of $NXZT$ on the plane (ABC) (Fig. 7). Then $\sigma_6^1=\sigma_{NTZ^1X^1}=a^2\dfrac{n}{k}$; $\sigma_6=\sigma_{NXZT}=-\dfrac{a^2}{km}\sqrt{m^2n^2+m^2k^2+n^2k^2}$. In the same way one can prove all the formulae in (3).

Theorem 2. *For any $k \neq 0$ the area function is a smooth function.*

Theorem 3. *For any $k \neq 0$ the area function is a differentiable function (exactly once).*

By differentiating we get:

$$\sigma_1^{(1)}(k) = -\frac{a^2mn^2}{k^2\sqrt{k^2m^2+k^2n^2+m^2n^2}};$$

$$\sigma_2^{(1)}(k) = \frac{a^2}{k^2}\left(\frac{1}{n}+\frac{1}{k}-1\right)\sqrt{m^2n^2+m^2k^2+n^2k^2}+$$

$$+\frac{a^2}{2}\left(\frac{2}{km}-\left(1-\frac{1}{m}-\frac{1}{n}-\frac{1}{k}\right)^2\right)\frac{k(m^2+n^2)}{\sqrt{m^2n^2+m^2k^2+n^2k^2}};$$

$$\sigma_3^{(1)}(k) = \frac{a^2k(m^2+n^2)(2mn-2m-n)}{2m^2n\sqrt{k^2m^2+k^2n^2+m^2n^2}};$$

$$\sigma_4^{(1)}(k) = -\frac{a^2k(m^2+n^2)(2m-1)}{2m^2\sqrt{m^2n^2+m^2k^2+n^2k^2}};$$

$$\sigma_5^{(1)}(k) = -\frac{a^2}{2m^2}\left[\left(-\frac{2}{k^2}+\frac{2m(k-1)}{k^3}\right)\sqrt{m^2n^2+m^2k^2+n^2k^2}+\right.$$

$$\left.+\left(2+m\frac{(k-1)^2}{k}\right)\frac{(m^2+n^2)}{\sqrt{m^2n^2+m^2k^2+n^2k^2}}\right];$$

$$\sigma_6^{(1)}(k) = \frac{a^2mn^2}{k^2\sqrt{m^2n^2+m^2k^2+n^2k^2}}.$$

It if easy to prove that:

1) $\sigma_1^{(1)}(k) > 0$ for any $k \in (-\infty, k_0]$, i.e. $\sigma(k)$ is a monotone increasing function in this interval.

2) $\sigma_3^{(1)}(k) < 0$ for any $k \in \left[\frac{n}{n-1}, 0\right)$, i.e. $\sigma(k)$ is a monotone decreasing function in this interval.

3) $\sigma_4^{(1)}(k) > 0$ for any $k \in \left(0, \frac{m}{m-1}\right]$, i.e. $\sigma(k)$ is a monotone increasing function in this interval.

4) $\sigma_6^{(1)}(k) > 0$ for any $k \in [1, +\infty)$, i.e. $\sigma(k)$ is a monotone decreasing function in this interval.

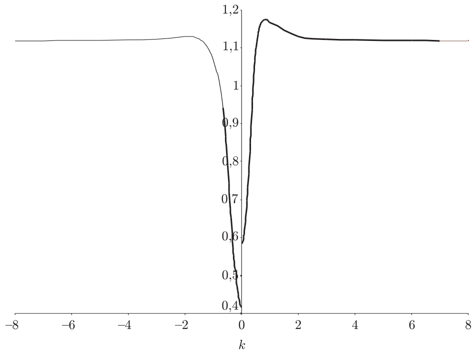
We prove also the following assertions:

I. $\sigma_2^{(1)}(k_0) > 0$, $\sigma_3^{(1)}\left(\frac{n}{n-1}\right) < 0$, it means that the area function has at least one local maximum in the interval $\left(k_0, \frac{n}{n-1}\right)$.

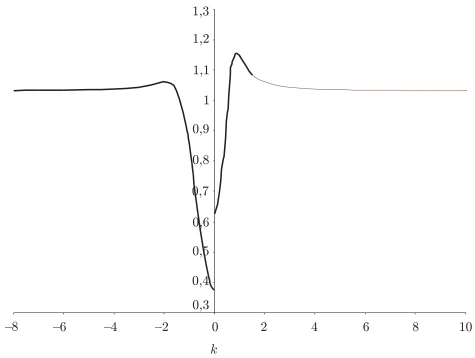
II. $\sigma_5^{(1)}\left(\frac{m}{m-1}\right) > 0$, $\sigma_6^{(1)} < 0$, it means that the area function has at least one local maximum in the interval $\left(\frac{m}{m-1}, 1\right)$.

Many special cases for $m < 0$ and $0 < n < 1$ show that the area function has only one maximum in these intervals. For example:

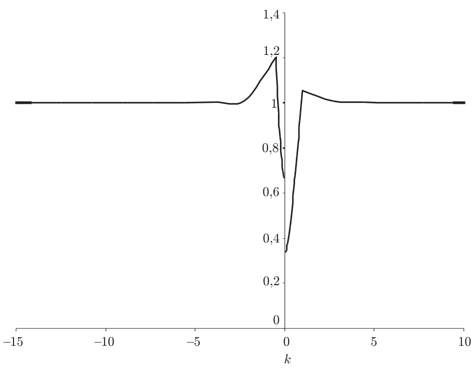
The graph of the area function when $m = 2/3$, $n = 1/3$



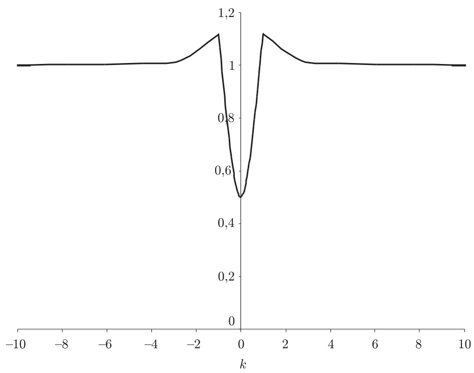
The graph of the area function when $m = -2$, $n = 1/2$



The graph of the area function when $m = -100$, $n = 1/3$



The graph of the area function when $m = -200$, $n = 1/2$



Remark. The last two assertions are not valid when $m \to -\infty$. Indeed in such a case: $\lim\limits_{m\to-\infty} k_0 = \dfrac{n}{n-1}$ and $\lim\limits_{m\to-\infty} \dfrac{m}{m-1} = 1$, i.e. the intervals $\left(k_0, \dfrac{n}{n-1}\right)$ and $\left(\dfrac{m}{m-1}, 1\right)$ are degenerated and there are no functions $\sigma_2(k)$ and $\sigma_5(k)$. Then:

$$\lim_{k\to n_0, k<n_0} \sigma_1^{(1)}(k) = -\frac{a^2(n-1)^3}{n\sqrt{1+(n-1)^2}} > 0, \qquad \lim_{k\to n_0, k<n_0} \sigma_3^{(1)}(k) = \frac{a^2(n-1)}{n\sqrt{1+(n-1)^2}} < 0,$$

where $n_0 = \dfrac{n}{n-1}$.

$$\lim_{k\to 1, k<1} \sigma_4^{(1)}(k) = \frac{a^2}{\sqrt{n^2+1}} > 0, \qquad \lim_{k\to 1, k>1} \sigma_6^{(1)}(k) = -\frac{a^2(n-1)^2}{\sqrt{n^2+1}} < 0.$$

It follows that the area function $\sigma(k)$ at the points $k = \dfrac{n}{n-1}$ and $k = 1$ is not differential.

From the last figure one can see that the area function has a minimum. This can be proved, namely: $\lim\limits_{k\to 0, k<0} \sigma_3^{(1)}(k) = \lim\limits_{k\to 0, k>0} \sigma_4^{(1)}(k)$, when n tends to $\dfrac{1}{2}$ and m tends to minus infinity.

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Albena Jordanova Botseva
18 Secondary School
Sofia, Bulgaria
e-mail::Albena_jordanova@mailcity.com

Victoria Petrova Daneva
Classic College of Fine Mechanics and Optics
Sofia, Bulgaria

КУБИЧНИ СЕЧЕНИЯ С ПОДВИЖНА РАВНИНА

Албена Йорданова, Виктория Данева

Разглеждаме сеченията на куб с подвижна равнина, зависеща от три параметъра. Доказваме главната теорема за сеченията и намираме функцията лице на сечението (Теорема 2). Показваме, че функцията лице е непрекъсната и диференцируема (от първи ред). В много разгледани частни случаи, с помощта на компютърно базирани методи, сме установили, че тази функция има поне два максимума и четири инфлексни точки.
Смятаме, че някои частни случаи могат да бъдат разгледани в часовете по математика в училище.