WEYL SPACES OF COMPOSITIONS

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Three Weyl spaces of nonorthogonal compositions of two base manifolds are investigated in the present paper. The hybridian tensor generated by the first Weyl space and introduced by G. Timofeev is chosen as a fundamental tensor for the second of them and the third Weyl space is conformal to the second one. Necessary and sufficient conditions, so that these three Weyl spaces to be spaces of compositions are found with the help of the prolonged covariant differentiation. The relations between the tensors of curvatures of these three Weyl spaces of compositions are established. The characteristics for an isotropic composition in these spaces are obtained.

1. Preliminary. Let $W_N(g_{\alpha\beta}, T_\sigma)$ be Weyl space with a fundamental tensor $g_{\alpha\beta}$ and a complementary covector $T_\sigma$.

After renormalization of the fundamental tensor by the law

$$\tilde{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta},$$

where $\lambda$ is a point function, the complementary covector $T_\sigma$ transforms by the law ([1], p.152)

$$\tilde{T}_\sigma = T_\sigma + \partial_\sigma \ln \lambda.$$

According to ([1], p.152) the fundamental tensor $g_{\alpha\beta}$ and the complementary covector $T_\sigma$ satisfy the identities

$$\nabla_\sigma g_{\alpha\beta} = 2T_\sigma g_{\alpha\beta}, \quad \nabla_\sigma g^{\alpha\beta} = -2T_\sigma g^{\alpha\beta}$$

where $g^{\alpha\beta}$ is the reciprocal tensor to $g_{\alpha\beta}$ and

$$g_{\alpha\beta} g^{\alpha\sigma} = \delta_\sigma^\beta.$$

The equalities

$$\nabla_\sigma \tilde{g}_{\alpha\beta} = 0, \quad \nabla_\sigma \tilde{g}^{\alpha\beta} = 0,$$

where with $\nabla$ is denoted the prolonged covariant differentiation, are introduced in [8].

Consider in the space $W_N$ the composition $X_n \times X_m$ of two base manifolds $X_n$ and $X_m$, $(n + m = N)$. The spaces $W_N$ of compositions will be denoted $W_N(X_n \times X_m)$. Two
positions \( P(X_n) \) and \( P(X_m) \) of the base manifolds pass through any point of the space \( WN(X_n \times X_m) \).

According to [2] and [3] each composition is completely definite with the field of the affinor \( a_\alpha^\beta \), satisfying the condition

\[
a_\alpha^\beta a_\alpha^\beta = \delta_\beta^\alpha.
\]

The affinor \( a_\alpha^\beta \) is called an affinor of the composition.

We presuppose that the composition is always integrable and nonorthogonal. The condition for integrability of the structure characterizes with the equality [6] and

\[
a_\beta^\sigma \nabla_{[\alpha} a_\sigma^\nu] - a_\alpha^\sigma \nabla_{[\beta} a_\sigma^\nu] = 0.
\]

According to [6, 7] the projecting affinors \( n_\alpha^\beta \), \( m_\alpha^\beta \) are defined by the equalities

\[
n_\alpha^\beta = \frac{1}{2} (\delta_\alpha^\beta + a_\alpha^\beta), \quad m_\alpha^\beta = \frac{1}{2} (\delta_\alpha^\beta - a_\alpha^\beta).
\]

It is true the equality \( g_{\alpha\beta} = n_{\alpha\beta} + 2a_{\alpha\beta} + m_{\alpha\beta} \), where

\[
G_{\alpha\beta} = \frac{n^\sigma}{\alpha} a_{\sigma\beta} g_{\sigma\nu}, \quad G_{\alpha\beta} = \frac{m^\sigma}{\alpha} a_{\sigma\beta} g_{\sigma\nu}, \quad G_{\alpha\beta} = a_{(\alpha}^\sigma a_{\beta)}^\nu g_{\sigma\nu}.
\]

The tensor \( G_{\alpha\beta} \) is named hybridian metrical tensor [4]. Due of (5) \( G_{\alpha\beta} \neq 0 \) for nonorthogonal compositions \( X_n \times X_m \).

According to [4] the tensor of the composition

\[
a_{\alpha\beta} = a_\alpha^\beta g_{\sigma\beta}
\]

and the fundamental tensor of \( WN \) satisfy the equation

\[
a_{\alpha\sigma} g^{\sigma\nu} a_{\nu\beta} = g_{\alpha\beta}.
\]

In [4] Timofeev has proved the following theorem

**Theorem 1.** The Weyl space \( WN(g_{\alpha\beta}, T_\sigma) \) is a space of composition \( WN(X_n \times X_m), \)
if and only if there exists a tensor \( a_{\alpha\beta} \), satisfying equalities (11) and

\[
g^{\sigma\rho} a_{\beta\rho} \nabla_{[\alpha a_{\sigma}]\nu} - g^{\sigma\rho} a_{\alpha\rho} \nabla_{[\beta a_{\sigma}]\nu} - 2T_{[\alpha} g_{\beta]\nu} - 2T_{\sigma} g^{\sigma\rho} a_{\alpha/\rho/} a_{\beta]\nu = 0.
\]

2. Weyl spaces of compositions. Let us note that by means of the prolonged covariant differentiation the equality (12) can be written in the form

\[
g^{\sigma\rho} a_{\beta\rho} \nabla_{[\alpha a_{\sigma}]\nu} - g^{\sigma\rho} a_{\alpha\rho} \nabla_{[\beta a_{\sigma}]\nu} = 0.
\]

214
Theorem 2. The Weyl space \( \overset{1}{W_N}(G_{\alpha\beta}, T_{\sigma}) \) is a space of composition \( \overset{1}{W_N}(X_n \times X_m) \), if and only if there exists a tensor \( a_{\alpha\beta} \), satisfying equalities

\[
a_{\sigma\alpha} \overset{0}{G}^{\sigma\nu} a_{\beta\nu} = G_{\alpha\beta},
\]

\[
G^{\sigma\rho} a_{\rho\beta} \overset{0}{1}\nabla_{[\alpha\nu/\sigma]} - G^{\sigma\rho} a_{\rho\alpha} \overset{0}{1}\nabla_{[\beta\nu/\sigma]} = 0.
\]

Proof. It is easily to see that the weight of the hybridian tensor \( G_{\alpha\beta} \) is \( \{2 \} \). Hence after renormalization (1) of \( g_{\alpha\beta} \) for \( G_{\alpha\beta} \) we have \( \overset{1}{G}_{\alpha\beta} = \lambda^2 G_{\alpha\beta} \). Since \( T_{\sigma} \) transforms by the law (2), then there exists only one Weyl space \( \overset{1}{W_N}(G_{\alpha\beta}, T_{\sigma}) \) with a fundamental tensor \( G_{\alpha\beta} \) and a complementary covector \( T_{\sigma} \), satisfying \( \overset{1}{1}\nabla_{\sigma} G_{\alpha\beta} = 2 T_{\sigma} G_{\alpha\beta} \).

According to [8]

\[
\overset{1}{1}\nabla_{\sigma} G_{\alpha\beta} = 0, \quad \overset{1}{1}\nabla_{\sigma} G^{\alpha\beta} = 0.
\]

Consider the tensor

\[
G^{\alpha\beta} = a_{\nu}^{(\alpha} a_{\beta)} g^{\sigma\nu}.
\]

Using (6) and (7) we find \( G_{\alpha\beta} G^{\alpha\beta} = G_{\delta\beta}^{\delta\beta} \), which means that \( G^{\alpha\beta} \{2 \} \) is a reciprocal tensor of the tensor \( G_{\alpha\beta} \).

Let the Weyl space \( \overset{1}{W_N}(G_{\alpha\beta}, T_{\sigma}) \) be a space of composition \( \overset{1}{W_N}(X_n \times X_m) \). Consider the tensor of the composition \( X_n \times X_m \)

\[
a_{\alpha\beta} = a_{\nu}^{\alpha} G_{\beta\nu}.
\]

Obviously \( a_{\alpha\beta} \{2 \} \). Taking into account (9), (10), (18) we obtained

\[
a_{\alpha\beta} = a_{\beta\alpha}.
\]

To prove (14) it is sufficient to substitute \( G^{\alpha\beta} \) and \( a_{\alpha\beta} \) from (17) and (19) in (14) and keeping in mind (4) and (18).

After prolonged covariant differentiation of (18) and applying (16) and (19) we find

\[
\overset{1}{1}\nabla_{\nu} G_{\beta\alpha} = \nabla_{\nu} a_{\nu}^{\alpha} G_{\beta\alpha}.
\]

Now after substitution of \( \nabla_{\nu} a_{\nu}^{\alpha} = \overset{1}{1}\nabla_{\nu} G_{\beta\alpha} G^{\sigma\beta} \) in (7) we obtain (15).

Theorem 3. The Weyl space \( \overset{2}{W_N}(g_{\alpha\beta}, a_{\alpha\beta} T_{\sigma}) \) is a space of composition \( \overset{2}{W_N}(X_n \times X_m) \), if and only if there exists a tensor \( a_{\alpha\beta} \), satisfying equalities (11) and

\[
g^{\sigma\rho} a_{\rho\beta} \overset{2}{1}\nabla_{[\alpha\nu]\sigma} - g^{\sigma\rho} a_{\alpha\rho} \overset{2}{1}\nabla_{[\beta\nu]\sigma} = 0.
\]

Proof. Let us consider the covector \( P_{\alpha} = a_{\alpha\beta} T_{\beta} \) and the equality

\[
\overset{2}{1}\nabla_{\sigma} g_{\alpha\beta} = 2 P_{\sigma} g_{\alpha\beta}.
\]
From (1) and (21) it follows \[ 2\nabla_{\sigma} \tilde{g}_{\alpha\beta} = 2(\partial_{\sigma} \lambda + \alpha_{\sigma}^{\nu} P_{\nu}) \tilde{g}_{\alpha\beta}, \] which means that \( P_{\sigma} \) transforms by the law (2). Hence there exists only one weyl space \( ^{2}W_{N}(g_{\alpha\beta}, P_{\sigma}) \) with a fundamental tensor \( g_{\alpha\beta} \) and a complementary covector \( P_{\sigma} \).

According to [8] the equality (21) takes the form

\[ \nabla_{\sigma} \tilde{g}_{\alpha\beta} = 0. \]  (22)

Let the Weyl space \( ^{2}W_{N}(g_{\alpha\beta}, P_{\sigma}) \) be a space of composition \( ^{2}W_{N}(X_{n} \times X_{m}) \). It is easy to see that the tensor (10) is the tensor of the composition \( X_{n} \times X_{m} \) and it satisfies (11).

From (7), (10), (22) it follows (20).

3. Transformation of Weyl connectedness in \( W_{N}(X_{n} \times X_{m}) \).

**Theorem 4.** The tensors of curvatures \( R_{\alpha\beta\gamma}^{\quad \sigma}, \quad \frac{1}{2} R_{\alpha\beta\gamma}^{\quad \sigma}, \quad \frac{2}{3} R_{\alpha\beta\gamma}^{\quad \sigma} \) of the spaces \( W_{N}(X_{n} \times X_{m}), \quad W_{N}(X_{n} \times X_{m}), \quad W_{N}(X_{n} \times X_{m}) \), respectively satisfy the equalities:

\[ \frac{1}{2} R_{\alpha\beta\gamma}^{\quad \sigma} = R_{\alpha\beta\gamma}^{\quad \sigma}, \]  (23)

\[ \frac{2}{3} R_{\alpha\beta\gamma}^{\quad \sigma} - \frac{1}{2} R_{\alpha\beta\gamma}^{\quad \sigma} = 4\nabla_{[\alpha} m^{\beta]} T_{\gamma]} T_{\sigma}, \]  (24)

where

\[ m T_{\beta} = m T_{\beta}. \]  (25)

**Proof.** From \( \nabla_{\alpha} T_{\beta} - \nabla_{\beta} T_{\alpha} = -T_{\alpha\beta}^{\quad \gamma} T_{\gamma} \) where \( T_{\alpha\beta}^{\quad \gamma} \) is the affine strain tensor, we obtain

\[ \nabla_{[\alpha} T_{\beta]} = \nabla_{[\alpha} T_{\beta]} \]  (26)

According to ([1], p.157) we have

\[ R_{\alpha\beta\gamma}^{\quad \sigma} = -2\nabla_{[\alpha} T_{\beta]}, \quad \frac{1}{2} R_{\alpha\beta\gamma}^{\quad \sigma} = -2\nabla_{[\alpha} T_{\beta]}, \]  (27)

From (26), (27) it follows (23).

Obviously there exists a conformal mapping between the spaces \( W_{N}(X_{n} \times X_{m}), \quad W_{N}(X_{n} \times X_{m}) \). According to ([1], p.157) the tensors of curvatures of the spaces \( W_{N}(X_{n} \times X_{m}), \quad W_{N}(X_{n} \times X_{m}) \) satisfy the equality

\[ \frac{2}{3} R_{\alpha\beta\gamma}^{\quad \sigma} = R_{\alpha\beta\gamma}^{\quad \sigma} + 2P^{\alpha}_{\gamma}[\beta P_{\alpha\nu}], \]  (28)

where

\[ 2P^{\alpha}_{\gamma}[\beta P_{\alpha\nu}] = \delta_{\beta}^{\gamma} \delta_{\alpha}^{\nu} + \delta_{\alpha}^{\nu} \delta_{\beta}^{\gamma} - \delta^{\alpha\nu} g_{\gamma\beta}, \]  (29)

\[ p_{\alpha\nu} = \nabla_{\alpha} p_{\nu} - \frac{1}{2} P^{\beta}_{\gamma}[\alpha P_{\beta\sigma}], \]  (30)
From (8), (25) and (31) we obtain

\[ p_\alpha = T_\alpha - P_\alpha. \]

Taking into account (30) and \( p_\alpha = 2 T_\alpha \), \( P^\nu = 0 \) we find

\[ p_\alpha = 2 m \sigma_\alpha T_\sigma = 2 m T_\alpha, \]

i.e. \( p_\alpha \in P(X_m) \).

At last from (28) and (32) it follows the validity of (24).

### 4. Isotropic composition in \( W_N(g_{\alpha\beta}, T_\sigma) \)

It is known that the Weyl space \( W_N(g_{\alpha\beta}, T_\sigma) \) assumes an isotropic composition if and only if the tensor of the composition \( a_{\alpha\beta} \) is asymmetric [4]. In this case the positions \( P(X_n) \) and \( P(X_m) \) have the same dimension and the space \( W_N(g_{\alpha\beta}, T_\sigma) \) is even-dimensional.

**Theorem 5.** The Weyl space \( W_N(g_{\alpha\beta}, T_\sigma) \) assumes an isotropic composition if and only if for any field of directions \( v^\alpha \) the fields of directions \( v^\alpha \) and \( a_{\alpha}^\nu v^\sigma \) are orthogonal in \( W_N(g_{\alpha\beta}, T_\sigma) \).

**Proof.** The fields of directions \( v^\alpha \) and \( a_{\alpha}^\nu v^\sigma \) are orthogonal in \( W_N(g_{\alpha\beta}, T_\sigma) \) if and only if

\[ g_{\alpha\beta} v^\alpha a_{\alpha}^\nu v^\sigma = 0. \]

According to (10) the equality (33) is equivalent to the equality \( a_{\alpha\beta} v^\alpha v^\beta = 0 \). Since \( v^\alpha \) are arbitrary, then (33) is equivalent to \( a_{(\alpha\beta)} = 0 \), i.e. the tensor of the composition is asymmetric.

**Corollary 1.** If one of the spaces of compositions \( W_N(X_n \times X_m), W_N(X_n \times X_m), W_N(X_n \times X_m) \) assumes an isotropic composition then the other two spaces assume isotropic compositions too.

The validity of the corollary follows from the fact that the tensors of the compositions of the spaces \( W_N(X_n \times X_m), W_N(X_n \times X_m), W_N(X_n \times X_m) \) are \( a_{\alpha\beta}, a_{\beta\alpha}, a_{\alpha\beta} \), respectively.

**REFERENCES**


217
ВАЙЛОВИ ПРОСТРАНСТВА ОТ КОМПОЗИЦИИ

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Нека е дадено вайлово пространство от неортогонална композиция. Разглеждат се още две вайлови пространства, свързани с даденото по следния начин: основният тензор на едното е въведен от Тимофеев хибриден тензор, а другото вайлово пространство е конформно на даденото. С помощта на продължено ковариантно диференциране са намерени необходими и достатъчни условия три те вайлови пространства да бъдат пространства от композиция и са намерени връзки между тензорите на кривината на тези пространства. Получена е характеристика за изотопна композиция в тези пространства.