CONFORMAL TRANSFORMATION OF SPECIAL COMPOSITIONS IN A THREE-DIMENSIONAL WEYL SPACE

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Special compositions, generated by a net in a space with a symmetric linear connection are considered in [6]. In [6] it is also introduced the prolonged covariant differentiation of satellites of the metric tensor of a Weyl space. In this paper, special compositions generated by a net in a 3-dimensional Weyl space are studied. Conformal geometry of special compositions in a 3-dimensional Weyl space is considered. It is proven, that an orthogonal Cartesian composition exists only in a 3-dimensional Riemannian space, where the form of the curvature tensor is found.

1. Preliminaries. Let $W_3 (g_{ij}, \omega_k)$ be a 3-dimensional Weyl space with a metric tensor $g_{ij}$ and a complementary vector $\omega_k$. The coefficients of the Weyl connection $\nabla$ are determined by the equation: $\Gamma^i_{ij} = \{ k \}_{ij} - (\omega_i \delta^k_j + \omega_j \delta^k_i - g_{ij} g^{ks} \omega_s)$, where $\{ k \}_{ij}$ are the Cristoffel symbols, determined by $g_{ij}$, $\det(g_{ij}) \neq 0$. The following equations are valid: $\nabla_k g_{ij} = 2 \omega_k g_{ij}$, $\nabla_k g_{ij} = -2 \omega_k g_{ij}$ [9]. Following [7], the prolonged covariant differentiation $\overset{\circ}{\nabla}$ of the satellite $A$ with weight $\{ p \}$ in the Weyl space is defined by $\overset{\circ}{\nabla}_i A = \nabla_i A - p \omega_i A$.

Let $(v_1, v_2, v_3)$ be a net in $W_3$, defined by the independent tangent vector fields $v^v_k$ of the curves of the net $(k = 1, 2, 3)$. We determine the inverse covectors $v^k_i$ of $v^v_k$ $(k = 1, 2, 3)$, respectively, by the equations:

$$(1.1) \quad v^k_i v^v_k = \delta^s_i \quad \Leftrightarrow \quad v^k_i v^v_k = \delta^s_i.$$

In the paper [7] there are found the derivative equations:

$$(1.2) \quad \overset{\circ}{\nabla}_k v^v_s = T^m_k v^v_m, \quad \overset{\circ}{\nabla}_k v^v_s = -T^m_k v^v_m, \quad k = 1, 2, 3.$$

Later we will consider a net $(v_1, v_2, v_3)$ in $W_3$, for which the independent tangent vector fields $v^v_k$ are normalized by the terms [9]:

$$(1.3) \quad g_{ij} v^v_i v^v_j = g_{ij} v^v_i v^v_j = g_{ij} v^v_i v^v_j = 1, \quad \cos \omega_s = g_{ks} g^{ks}.$$
where $\omega = \omega$ are the angles defined by $v$ and $v$, $s, k = 1, 2, 3$, $s \neq k$. In the paper [4] the following relations are given:

$$g_{ik}v^k = \cos \omega \frac{1}{s_1}v_i + \cos \omega \frac{2}{s_2}v_i + \cos \omega \frac{3}{s_3}v_i, \quad s = 1, 2, 3.$$  

The net $(v, v, v) \in W_3$, for which conditions (1.3) and (1.4) are valid, will be called normalized. Let us remark that normalized nets $(v, v, v) \in W_3$ are studied in the paper [1]. According to [4, Lemma 1.1] for the coefficients of equations (1.2) the following equations are valid:

$$\cos \omega \frac{1}{s_1}T_k + \cos \omega \frac{2}{s_2}T_k + \cos \omega \frac{3}{s_3}T_k = 0, \quad \cos \omega \frac{m}{s}T_i + \cos \omega \frac{m}{sm}T_k = \partial_i \cos \omega, \quad k, s = 1, 2, 3.$$ 

Let us take a given composition $X_2 \times X_1$ in $W_3$, where $X_2$ (dim $X_2 = 2$) and $X_1$ (dim $X_1 = 1$) are the fundamental manifolds of the composition. Then through each point $p \in W_3$ there exists exactly one position $P(X_2)$ and $P(X_1)$, from $X_2$ and $X_1$ respectively. Following [5], $W_3$ is a space of composition $W_3(X_2 \times X_1)$, provided there exists a tensor field $\alpha^i$ of type $(1, 1)$ for which are valid the following equations:

$$\alpha^i_1 \alpha^j_2 = \delta^k_i,$$

and the condition for integration of the structure $\alpha^i$. According to [5], the Nijenhuis tensor $N^i_{ij}$ for $\alpha^i_1$ is annulled, i.e. $N^i_{ij} = \alpha^i_1 \nabla_a \alpha^j_2 - \alpha^j_2 \nabla_a \alpha^i_1 - \alpha^k_1 \nabla_{(a} \alpha^i_1 \nabla_{b)} \alpha^j_2 - \delta_{ij} \alpha^k_1 \nabla_a \alpha^k_2 = 0$.

In [6] is defined the affinor $\alpha^i_1$ of the composition in the Weyl space $W_n$. In $W_3$ for affinor $\alpha^1_0$, determined uniquely by the net $(v, v, v)$, are realized the following conditions:

$$\alpha^i_1 = \frac{v^i_1}{s_1}v_1 + \frac{v^i_2}{s_2}v_2 - \frac{v^i_3}{s_3}v_3 = \delta^i_1 - \frac{3}{s_3}v^i_3, \quad \alpha^i_2 = \frac{v^i_2}{s_2}, \quad \alpha^i_3 = \frac{v^i_3}{s_3} = -v^i_3.$$ 

Let $\tau$ be a conformal transformation of $W_n (g_{ij}, \omega_k)$ into $\overline{W}_n (\overline{g}_{ij}, \overline{\omega}_k)$. Then following [9], in the corresponding points of these spaces we have: $\overline{g}_{ij} = g_{ij}, \overline{\omega}_k = \omega_k - p_i$, where the covector $\tau_i$ is called the vector of the conformal transformation $\tau$.

Let $\Gamma^a_{ij}$ and $\Gamma^b_{ik}$ be the coefficients of the Weyl connections of $\overline{W}_3$ and $W_3$, respectively. Then we have [9]: $\Gamma^d_{ij} = \Gamma^d_{ij} + \delta^d_i p_j + \delta^d_j p_i - g_{ij}g^{ks}p_s$.

Let $W_3$ and $\overline{W}_3$ be conformally equivalent Weyl spaces. Then with respect of the connection $\nabla$ of $W_3$ the derivative equations give the expression of (1.2), while with respect of the connection $\overline{\nabla}$ of $\overline{W}_3$ they have the following form:

$$\overline{\nabla}v^i = \overline{P}_{m}^i v^m, \quad \overline{\nabla}_i v_k = -\overline{P}_{i}^m v_m, \quad k, l = 1, 2, 3.$$ 

The relation between the coefficients in (1.2) and (1.8) in the case of conformal transformation $\tau$ is found in [8], i.e.

$$\overline{P}_{s}^i = \frac{l}{s}T_s + p_m v^m \frac{l}{s}v_k - p_e g_{km} v^m, \quad p_l = \frac{g_{sk} p_s v_k}{s}.$$ 

where $p^l = g_{sk} p_s v_k$, $s = 1, 2, 3$, $l = 1, 2, 3$.

The vector of the conformal transformation $\tau_i$ has the form:

$$p_j = \left( p_m v^m \right)^{\frac{1}{s}}v_j + \left( p_m v^m \right)^{\frac{2}{s}}v_j + \left( p_m v^m \right)^{\frac{3}{s}}v_j,$$

We can assert that when we have a conformal transformation $\tau$ of compositions $X_2 \times X_1$
$X_1 \in W_3$ and $X_2 \times X_1 \in \overline{W}_3$, associated with the normalized net $(v, v, v)$, conditions (1.5) are valid about $\frac{t}{s}$ and their analogous equations about $\frac{p}{s}$.

2. Conformal transformation of a composition in $W_3$. Following [10], the composition $X_2 \times X_1 \in W_3$ is called geodesic-Chebyshevian, if the tangent section of $P(X_2)$ and the tangent vector of the curve $P(X_1)$ can be translated parallelly in the direction of every curve of $P(X_2)$.

The composition $X_2 \times X_1 \in W_3$ is called Chebyshevian-geodesic, if the tangent section of $P(X_2)$ is translated parallelly in the curve $P(X_1)$, and the tangent vector of $P(X_1)$ is translated parallelly in the curve $P(X_1)$, i.e. the curve $P(X_1)$ is geodesic.

Definition 2.1. A composition $X_2 \times X_1 \in W_3$ is called conformally geodesic-Chebyshevian (respectively conformally Chebyshevian-geodesic) when it can be transformed into a geodesic-Chebyshevian (respectively Chebyshevian-geodesic) composition $X_2 \times X_1 \in \overline{W}_3$ by the transformation $\tau$.

In [3, Theorem 1, Theorem 3, Theorem 4] are found geometrical characteristics and conditions for geodesic-Chebyshevian and Chebyshevian-geodesic compositions, i.e.

1) If $X_2 \times X_1 \in \overline{W}_3$ is a geodesic-Chebyshevian composition, then according to [3, Theorem 1], we obtain:

$$P_1^{3} = \frac{1}{3} P_2^{3}, P_1^{2} = \frac{2}{3} P_2^{2}, P_1^{1} = \frac{3}{3} P_2^{1}, P_1^{3} = \frac{3}{3} P_2^{3} = \frac{3}{3} P_2^{2} = 0.$$  

2) If $X_2 \times X_1 \in \overline{W}_3$ is a Chebyshevian-geodesic composition, then according to [3, Theorem 3, Theorem 4], we obtain:

$$P_1^{3} = P_2^{3}, P_1^{2} = P_2^{2} = P_1^{1} = P_2^{1} = \frac{3}{3} P_2^{3} = \frac{3}{3} P_2^{2} = 0, \quad \frac{3}{3} P_2^{1} = \frac{3}{3} P_2^{2} = \frac{3}{3} P_2^{3} = v^{s}, \quad s = 1, 2, 3.$$

Theorem 2.1. A composition $X_2 \times X_1 \in W_3$, determined by the normalized net $(v, v, v)$, is a conformally geodesic-Chebyshevian if and only if the following conditions are valid:

$$T_1^{3} = \cos \omega p_1^{1}, \quad T_1^{2} = \cos \omega p_2^{2}, \quad T_1^{1} = \cos \omega p_3^{3}, \quad T_2^{3} = \cos \omega p_1^{1}, \quad T_2^{2} = \cos \omega p_2^{2}, \quad T_2^{1} = \cos \omega p_3^{3}, \quad T_3^{3} = \cos \omega p_1^{1} - p_3^{3} m, \quad T_3^{2} = \cos \omega p_2^{2} - p_3^{3} m, \quad T_3^{1} = \cos \omega p_3^{3}.$$  

The vector $p_3^{k}$ of the conformal transformation $\tau$ satisfies the following condition:

$$2 p_1^{1} m^1 = p_1^{1} + \cos \omega p_2^{2} + T_1^{1} v^1, \quad p_1^{2} m^2 = \cos \omega p_2^{2} + p_2^{2} + T_2^{2} v^2,$$

$$p_3^{3} m^3 = - \left( T_3^{1} v^1 + T_3^{2} v^2 + T_3^{3} v^3 \right).$$  

Proof. Let $X_2 \times X_1 \in W_3$ be a conformally geodesic-Chebyshevian composition. Then for $X_2 \times X_1 \in \overline{W}_3$ equations (2.1) are valid. After contracting equation (1.9) to 134
the vectors $v^{k}$ and $v^{k}$, and having in mind (1.5) and (2.1), we obtain (2.3).

Conversely, if equations (2.3) are valid, then from (1.9) it follows (2.1). In order to determine the vector of the conformal transformation $p_{k}$ from (1.10), we use the functions $p_{1}^{1}, p_{2}^{2}$ and $p_{3}^{3}$ from (2.3). Equations (2.4) are obtained through a suitable transformation of (2.3). □

Analogously, using (1.5), (1.9) and (2.2), it follows that:

**Theorem 2.2** A composition $X_{2} \times X_{1} \in W_{3}$, determined by the normalized net $(v, v, v)$, is conformally Chebyshevian-geodesic if and only if the following conditions are valid:

\[
\begin{vmatrix}
\frac{1}{2} & T_{k}^{1} \xi^k & = p_{1}^{1} & T_{k}v^k = p_{1}^{1} & T_{k}v^k = \cos \omega p_{1}^{1}, & \frac{2}{3} & T_{k}v^k = \cos \omega p_{1}^{2}, \\
\frac{1}{3} & T_{k}v^k = p_{2}^{2}, & T_{k}v^k = p_{2}^{2}, & T_{k}v^k = \cos \omega p_{2}^{2}, & T_{k}v^k = \cos \omega p_{2}^{3}, \\
\frac{1}{3} & T_{k}v^k = p_{3}^{3}, & T_{k}v^k = p_{3}^{3}, & T_{k}v^k = \cos \omega p_{3}^{2}, & T_{k}v^k = \cos \omega p_{3}^{3}.
\end{vmatrix}
\]

(2.5)

The vector $p_{k}$ of the conformal transformation $\tau$ has the form (1.10), where the coefficients $p_{m}^{n}$, $s = 1, 2, 3$, are defined in (2.5).

3. Orthogonal compositions in $W_{3}$.

**Definition 3.1** [11]. A composition $X_{2} \times X_{1} \in W_{3}$ is orthogonal, when the vectors $v^{k}, v^{k}$ and $v^{k}$ are orthogonal, i.e.

\[
\begin{align*}
g_{ij}v_{1}^{i}v_{1}^{j} &= g_{ij}v_{2}^{i}v_{2}^{j} = 0 \iff \cos \omega_{13} &= \cos \omega_{23} = 0.
\end{align*}
\]

In the paper [11] it is introduced a tensor of type (0,2):

(3.2) $a_{ij} = a_{ij}^{k}g_{kj}$.

which is called the tensor on the composition in $W_{n}$. It is proved, that a composition in $W_{n}$ is orthogonal if and only if $a_{ij} = a_{ji}$. In this case, the tensor $a_{ij}$ is called associated with the metric tensor $g_{ij}$. From (3.2) it follows, that $a_{ij}$ is nondegenerate, i.e. $\det(a_{ij}) \neq 0$.

Let the composition $X_{2} \times X_{1} \in W_{3}$, determined by the normalized net $(v, v, v)$ be orthogonal. Using (1.6), (3.2) and $a_{ij} = a_{ji}$, for the metric tensor $g_{ij}$ we have the form:

(3.3) $a_{ij}^{k}a_{ij}^{k}g_{kk} = g_{ij}$.

According to [5], the condition (3.3) for the metric tensor means, that we can consider $W_{3}$ as a Riemannian space with the structure of an almost product $a_{ij}^{k}$ about the connection $\nabla_{i}$, determined by $g_{ij}$. Having in mind (1.1), (1.3), (1.7), (3.1), (3.2) and (3.3) we get:

**Lemma 3.1.** The Weyl space $W_{3}$ is a space of orthogonal composition $X_{2} \times X_{1}$, determined by the normalized net $(v, v, v)$ if and only if the metric tensor $g_{ij}$ and the tensor $a_{ij}$ associated with it, have the form:

(3.4) $g_{ij} = \frac{1}{2}v_{i}v_{j} + \frac{1}{2}v_{i}v_{j} + v_{i}v_{j} + \cos \omega_{12}(\frac{1}{2}v_{i}v_{j} + v_{i}v_{j})$, $a_{ij} = g_{ij} - \frac{3}{2}v_{i}v_{j}$.

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Taking into account (1.3), (3.1), (3.3) and (3.4), the tensor $g_{ij}$ in $W_3$ determines a positive definite metric (Riemannian), and the tensor $a_{ij}$ – an associated metric, which is indefinite with signature (2, 1).

**Theorem 3.1.** Let the conformally geodesic-Chebyshevian composition $X_2 \times X_1 \in W_3$ be orthogonal. Then the following relations are valid:

\[
T^i_k = 0, \quad T^1_k v^k = T^2_k v^k = T^3_k v^k = 0, \quad p_m v^m = 0.
\]

The metric tensor on $W_3$ has the form (3.4).

**Proof.** The equations (3.5) follow from (3.1), (3.4) and Theorem 2.1. □

Having in mind Theorem 2.2, (3.1) and (3.4), we get the following:

**Theorem 3.2.** Let the conformally Chebyshevian-geodesic composition $X_2 \times X_1 \in W_3$ be orthogonal. Then the following conditions are valid:

\[
\frac{2}{3} T^i_k = 0, \quad \frac{1}{3} T^1_k v^k = \frac{1}{2} T^2_k v^k = \frac{2}{3} T^3_k v^k = 0.
\]

The metric tensor of $W_3$ has the form (3.4), and the vector of the conformal transformation $\tau$ has the form:

\[
p_k = -\frac{3}{4} T^s_k v^s v^k - \frac{3}{2} T^s_j v^s v^j + \omega_{3, j} v^s v^k.
\]

Let the composition $X_2 \times X_1 \in W_3$ be orthogonal and Cartesian. Then, according to [2], for the affinor $a^s_k$ with respect to the Weyl connection $\nabla$, we have: $\nabla a^s_k = 0$. According to [9], the integrability condition for the last equation has the form:

\[
R^l_j a^s_l = R^s_l a^1_j.
\]

where $R^{s, l}_{j, k}$ is the curvature tensor for connection $\nabla$. In [2, Theorem 5] it is proven, that the curvature tensor of an arbitrary Weyl space $W_3$ has the following form:

\[
1 \frac{1}{3} \left\{ (g_{jk} S_{lm} - g_{lk} S_{jm}) g^{ms} + S_{jk} \delta^l_s - S_{ik} \delta^l_j + (S_{jl} - S_{ij}) \delta^l_k \right\},
\]

where $S_{jk} = 2R_{jk} + R_{kj} - \frac{3}{4} g_{jk} R$, $R_{jk}$ is the Ricci tensor, and $R = g^{jk} R_{jk}$ - the scalar curvature. Using (1.1) and (1.7), we can prove that the equation (3.8) is equivalent to:

\[
R^{s, l}_{j, k} - \frac{3}{4} T^s_k v^s v^k = R^s_l a^s_k.
\]

Substituting (3.9) in (3.10) and using a series of transformations, in view of (1.7), (3.3) and (3.4), we obtain:

\[
S_{jk} = S_{kj} = \frac{3R}{4} \left( g_{jk} - 2v^s v^k \right) = \frac{3R}{4} a_{j, k}, \quad R_{jk} = \frac{R}{2} \left( g_{jk} - \frac{3}{4} v^j v^k \right) = \frac{R}{4} (g_{jk} + a_{j, k}).
\]

According to (3.9) and the last equation, because of the symmetry of Ricci tensor, it follows that:

**Theorem 3.3.** Every $W_3$, containing an orthogonal Cartesian composition $X_2 \times X_1$, is a Riemannian space $V_3$. The curvature tensor of $V_3$ of type (0, 4) has the form:

\[
R^i_{j, k, l} = \frac{1}{3} \left( g_{jk} g_{il} - g_{ik} g_{jl} - v^s v^k g_{il} + v^s v^l g_{jk} - v^s v^i g_{jk} + v^s v^j g_{ik} \right),
\]

where $R^i_{j, k, l}$ is determined by the Riemannian connection $\tilde{\nabla}$ with components 136.
the Christoffel symbols \( \{ \Gamma^k_{ij} \} \).

Using (1.3), we immediately obtain the following:

**Corollary 3.1.** Let \( V_3 \) be a Riemannian space, containing the orthogonal Cartesian composition \( X_2 \times X_1 \). Then, for the Ricci curvatures in the direction of the net vectors \((v_1, v_2, v_3)\), the following equalities hold:

\[
R_{jk}^1 v_1^j v_1^k = R_{jk}^2 v_2^j v_2^k = \frac{R}{2}, \quad R_{jk}^3 v_3^j v_3^k = 0.
\]

**REFERENCES**


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КОНФОРМНА ТРАНСФОРМАЦИЯ НА СПЕЦИАЛНИ КОМПОЗИЦИИ В ТРИМЕРНО ВАЙЛОВО ПРОСТРАНСТВО

Добринка К. Грибачева

Специални композиции, породени от мрежа в пространство със симетрична линейна свързаност са изучавани в [6]. В [6] е въведено продължено ковариантно диференциране на сателитите на метричния тензор във Вайлово пространство. В тази статия изучаваме специални композиции, породени от мрежа в тримерно Вайлово пространство. Разгледана е конформна геометрия на специални композиции в тримерно Вайлово пространство. Доказано е, че ортогонално декартова композиция съществува само в тримерно Риманово пространство, където е намерен вида на тензора на кривина.