INVARIANT DISTANCES AND METRICS IN COMPLEX ANALYSIS

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The aim of this survey lecture is to present some research areas of the invariant distances and metrics theory. We discuss the main invariant objects and their relations to complex analysis.

1. Holomorphically contractible families of functions. An old method of complex analysis is to study properties of holomorphic mappings \( F : G \rightarrow D \) (\( F \in \mathcal{O}(G, D) \)) using appropriate pseudodistances. More precisely, we endow \( G \) and \( D \) with pseudodistances \( d_G \) and \( d_D \), respectively, in such a way that every holomorphic mapping \( F : G \rightarrow D \) is a contraction, i.e.

\[
(*) \quad d_D(F(a), F(z)) \leq d_G(a, z), \quad a, z \in G.
\]

We expect that, if the pseudodistances \( d_G \) and \( d_D \) are reasonably chosen, then they describe some properties of \( \mathcal{O}(G, D) \). This method, nowadays almost standard, has its roots in papers of Carathéodory from the twenties of the last century (cf. [3]). It has appeared that sometimes, instead of pseudodistances, one should consider more general objects, e.g. pluricomplex Green functions. In this way we are led to the following concept of holomorphically contractible families of functions:

Suppose that we are given a family \( \underline{d} = (d_G)_G \) of functions \( d_G : G \times G \rightarrow \mathbb{R}_+ \), where \( G \) runs over all (non-empty) domains in all \( \mathbb{C}^n \)'s (in fact, \( G \) can run over all connected complex manifolds or even over all connected complex analytic spaces). We say that \( \underline{d} \) is holomorphically contractible if the following two conditions are satisfied:

**Normalization** for the unit disc \( \mathbb{D} \subset \mathbb{C} \), the function \( d_\mathbb{D} \) coincides with the Möbius distance \( m_\mathbb{D} \), i.e.

\[
d_\mathbb{D}(a, z) = m_\mathbb{D}(a, z) := \frac{|z - a|}{1 - \bar{a}z}, \quad a, z \in \mathbb{D};
\]

**Contractibility** for any domains \( G \subset \mathbb{C}^n, \ D \subset \mathbb{C}^m \), every mapping \( F \in \mathcal{O}(G, D) \) satisfies (*)

In particular, if \( F : G \rightarrow D \) is biholomorphic, then \( d_D(F(a), F(z)) = d_G(a, z) \), \( a, z \in G \).

The main contractible families of functions are the following ones:

- **Möbius pseudodistance:**
  \[
e_G^*(a, z) := \sup\{m_\mathbb{D}(f(a), f(z)) : f \in \mathcal{O}(G, \mathbb{D})\} = \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}), \ f(a) = 0\};
\]
the function \( c_\gamma := \tanh^{-1} c_\gamma^* \) is called the Carathéodory pseudodistance.

- Higher order Möbius function:
  \[ m^{(k)}_\gamma(a, z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G, \mathbb{D}), \ \text{ord}_a f \geq k \} \quad (k \in \mathbb{N}), \]
  where \( \text{ord}_a f \) denotes the order of zero of \( f \) at \( a \).

- Pluricomplex Green function:
  \[ g_\mathcal{G}(a, z) := \sup\{u(z) : u : G \to [0, 1], \ \log u \in \mathcal{PSH}(G), \ \sup_{w \in G \setminus \{a\}} u(w)/\|w-a\| < +\infty\}, \]
  where \( \mathcal{PSH}(G) \) denotes the family of all functions plurisubharmonic on \( G \).

- Lempert function:
  \[ k_\gamma^*(a, z) := \inf\{m_D(\lambda, \mu) : \lambda, \mu \in \mathbb{D}, \exists \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : \varphi(\lambda) = a, \ \varphi(\mu) = z\} \]
  \[ = \inf\{\mu \in [0, 1] : \exists \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : \varphi(0) = a, \ \varphi(\mu) = z\}. \]

It is well known that \( c_\gamma = m_G^{(1)} \leq m_G^{(k)} \leq g_\mathcal{G} \leq k_\gamma^* \), and for any holomorphically contractible family \( (d_G) \) we have \( c_\gamma \leq d_G \leq k_\gamma^* \), i.e. the Möbius family is minimal and the Lempert family is maximal. The pseudodistance

\[ k_G := \sup\{d : d : G \times G \to \mathbb{R}_+ \text{ is a pseudodistance with } \tanh d \leq k_\gamma^*\} \]

is called the Kobayashi pseudodistance. Put \( k_G^* := \tanh k_G \). The family \((k_G^*)\) is holomorphically contractible in the sense of our definition.

2. Holomorphically contractible families of pseudometrics. Parallel to the category of holomorphically contractible families of functions one studies holomorphically contractible families of pseudometrics. Suppose we are given a family \( \delta = (\delta_G) \) of pseudometrics \( \delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+ \),

\[ \delta_G(a; \lambda X) = |\lambda|\delta_G(a; X), \quad (a, X) \in G \times \mathbb{C}^n, \ \lambda \in \mathbb{C}, \]

where \( G \) runs over all domains \( G \subset \mathbb{C}^n \). We say that \( \delta \) is holomorphically contractible if the following two conditions are satisfied:

- \((\text{Normalization})\) \( \delta_\mathbb{D}(a; X) = \gamma_\mathbb{D}(a; X) := \frac{|X|}{1 - |a|^2}, \quad (a, X) \in \mathbb{D} \times \mathbb{C}; \)

- \((\text{Contractibility})\) for any domains \( G \subset \mathbb{C}^n \), \( D \subset \mathbb{C}^n \) and every mapping \( F \in \mathcal{O}(G, D) \), we have \( \delta_D(F(a); F'(a)(X)) \leq \delta_G(a; X), \quad (a, X) \in G \times \mathbb{C}^n \).

In particular, if \( F : G \to D \) is biholomorphic, then \( \delta_D(F(a); F'(a)(X)) = \delta_G(a; X) \), \( (a, X) \in G \times \mathbb{C}^n \).

The following holomorphically contractible families of pseudometrics correspond to the above holomorphically contractible families of functions.

- Carathéodory–Reiffen pseudometric:
  \[ \gamma_G(a; X) := \sup\{|f'(a)(X)| : f \in \mathcal{O}(G, \mathbb{D}), \ f(a) = 0\}. \]

- Higher order Reiffen pseudometric:
  \[ \gamma_G^{(k)}(a; X) := \sup\left\{ \frac{1}{k!} |f^{(k)}(a)(X)|^{1/k} : f \in \mathcal{O}(G, \mathbb{D}), \ \text{ord}_a f \geq k \right\} \quad (k \in \mathbb{N}). \]

- Azukava pseudometric:
  \[ A_G(a; X) := \limsup_{\zeta \to 0} \sup_{\zeta \neq 0} \frac{1}{|\lambda|} g_G(a, a + \lambda X). \]
• **Kobayashi–Royden pseudometric:**

$$\kappa_G(a; X) := \inf \{ \alpha \geq 0 : \exists \varphi \in O(D,G) : \varphi(0) = a, \alpha \varphi(0) = X \}.$$  

It is well known that $$\gamma_G = \gamma_G^{(1)} \leq \gamma_G^{(b)} \leq A_G \leq \kappa_G,$$ and for any holomorphically contractible family of pseudometrics $$(\delta_G)_G$$ we have $$\gamma_G \leq \delta_G \leq \kappa_G,$$ i.e. the Carathéodory–Reiffen pseudometric is minimal and the Kobayashi–Royden pseudometric is maximal.

3. **Problems.** Let us mention a few research areas which are important from the point of view of the theory.

• **Effective formulas.** The theory suffers from the lack of effective examples, i.e. examples of domains $G \subset \mathbb{C}^n$, for which all/some of basic contractible functions or pseudometrics can be effectively determined. In fact, even in the case $n = 1$, besides the unit disc and annuli, there are no examples. The only general class of domains for which effective formulas are known, are so-called *elementary Reinhardt domains* of the form

$$\{(z_1, \ldots, z_n) : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < 1 \}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

• **Hyperbolicity.** Given a holomorphically contractible family of functions $d$ (resp. pseudometrics $\delta$) and a domain $G \subset \mathbb{C}^n$, find conditions under which $d_G(a, z) > 0$ for all $a, z \in G$, $a \neq z$ (resp. $\delta_G(a; X) > 0$ for all $(a, X) \in G \times \mathbb{C}^n$, $X \neq 0$). The question is essential only for unbounded domains because all bounded domains are hyperbolic with respect to all contractible families.

• **Completeness.** Given a holomorphically contractible family of distances $d$ and a domain $G \subset \mathbb{C}^n$, find conditions under which the metric space $(G, d_G)$ is (in certain sense) complete. Completeness is strictly connected with other holomorphic properties of $G$, e.g. with tautness.

• **Boundary behavior and localization.** Here we have a large variety of problems related to the asymptotic behavior of $d_G(a, z)$ when $z \longrightarrow b \in \partial G$, or $a, z \longrightarrow b \in \partial G$ (resp. of $\delta_G(a; X)$ when $a \longrightarrow b \in \partial G$). It has appeared that there are strong links between asymptotic behavior of certain contractible functions and boundary regularities of $G$ (like strong pseudoconvexity or hyperconvexity).

• **Geodesics.** Given a holomorphically contractible family of functions $d$ (resp. pseudometrics $\delta$), a domain $G \subset \mathbb{C}^n$, and two points $a, b \in G$ (resp. a point $a \in G$ and a vector $X \in \mathbb{C}^n$), decide whether there exist a holomorphic mapping $\varphi : \mathbb{D} \longrightarrow G$ and points $\lambda, \mu \in \mathbb{D}$ such that $\varphi(\lambda) = a$, $\varphi(\mu) = b$, and $d_G(a, b) = m\overline{\varphi}(\lambda, \mu)$ (resp. a holomorphic mapping $\varphi : \mathbb{D} \longrightarrow G$, a point $\lambda \in \mathbb{D}$, and a number $\alpha \in \mathbb{C}$ such that $\varphi(\lambda) = a$, $\alpha \varphi(\lambda) = X$, and $\delta_G(a; X) = \gamma_G(\lambda; \alpha)$. The answer is always positive in the class of convex domains. Many other cases are still open.

• **Product property.** Given a holomorphically contractible family of functions $d$ (resp. pseudometrics $\delta$), we ask whether for all domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ we have

$$d_{G \times D}((a,b),(z,w)) = \max\{d_G(a, z), d_D(b, w)\}, \quad (a,b),(z,w) \in G \times D$$

(resp.

$$\delta_{G \times D}((a,b);(X,Y)) = \max\{\delta_G(a; X), \delta_D(b; Y)\}, \quad (a,b) \in G \times D, \quad (X,Y) \in \mathbb{C}^n \times \mathbb{C}^m.$$  

It has been proved that the families $\gamma_G^k$, $\overline{\delta}_{D}^k$, $\overline{\delta}_{D}^k$, $\overline{\gamma}_{D}^k$, $\overline{\gamma}_{D}^k$, $\overline{\delta}_{D}^k$ have the product property. It is also known that for $k \geq 2$ the families $\overline{w}(k)$, $\overline{\gamma}(k)$ have no product property.

• **Different contractibility conditions.** Our contractibility condition deals with all
holomorphic mappings $F : G \to D$. One can consider weaker conditions which require the contractibility only for some subclasses of holomorphic mappings, e.g. injective holomorphic mappings or even only biholomorphic mappings. The latter case allows, for instance, to consider Bergman pseudodistance and pseudometric as contractible objects.

- **Lempert theorem.** Let $\mathcal{L}_n$ be the class of all domains $G \subset \mathbb{C}^n$ for which $c^*_G \equiv k^*_G$ and $\gamma_G \equiv \infty$. In particular, if $G \in \mathcal{L}_n$, then all holomorphically contractible functions (reps. pseudometrics) coincide on $G$. It is clear that $\mathcal{L}_n$ is invariant under biholomorphic mappings. Moreover, if a domain $G \subset \mathbb{C}^n$ may be exhausted by domains from $\mathcal{L}_n$, then $G \in \mathcal{L}_n$.

The fundamental Lempert theorem ([10, 11]) says that all convex domains belong to $\mathcal{L}_n$. For more than 20 years the following conjecture was open. Any bounded pseudo-convex domain $G \in \mathcal{L}_n$ may be exhausted by domains biholomorphic to convex domains. The first counterexample was recently constructed in a series of papers by J. Agler, C. Costara, and N. J. Young [1–6] This is the so-called symmetrized bidisc

$$G_2 := \{(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{D}\}.$$ One should mention that for $n \geq 3$ the holomorphic geometry of the symmetrized $n$–disc

$$G_n := \{(\sigma_1(\lambda), \ldots, \sigma_n(\lambda)) : \lambda \in \mathbb{D}^n\},$$

where $\sigma_1, \ldots, \sigma_n$ are standard fundamental symmetric polynomials, remains still unclear.

4. Generalized holomorphically contractible families. The Möbius and Lempert functions are symmetric. The higher Möbius functions and the Green function are not symmetric in general. Their definitions distinguish one point (pole) at which we impose growth conditions. From that point of view it is natural to investigate objects with more general growth conditions. For instance, the Green function $g_G$ may be generalized as follows. Let $G \subset \mathbb{C}^n$ be a domain and let $p : G \to \mathbb{R}^+$ be an arbitrary function. Put $|p| := \{z \in G : p(z) > 0\}$ and define

$$g_G(p, z) := \sup\{u(z) : u : G \to [0, 1], \log u \in \mathcal{P}(G), \forall a \in |p| : \sup_{w \in G \setminus \{a\}} u(w)/\|w - a\|^{p(a)} < +\infty\}, \quad z \in G.$$ The function $g_G(p, \cdot)$ is called the **generalized pluricomplex Green function with poles (weights, pole function)** $p$.

In the case where $p = \chi_A$ is the characteristic function of a set $A \subset G$, we put $g_G(A, \cdot) := g_G(\chi_A, \cdot)$. Obviously, $g_G \{a, \cdot\} = g_G(a, \cdot)$. In the case where the set $|p|$ is finite, the function $g_G(p, \cdot)$ was first introduced by P. Lelong in [9]. The generalized pluricomplex Green function was recently intensively studied by many authors.

Using similar ideas, one can consider the Möbius function. Let $G \subset \mathbb{C}^n$ be a domain and let $p : G \to \mathbb{Z}_+$ be an arbitrary function. Define

$$m_G(p, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}), \forall a \in |p| : \text{ord}_a f \geq p(a)\}, \quad z \in G.$$ The function $m_G(p, \cdot)$ is called the **generalized Möbius function with weights** $p$. Similarly as in the case of the generalized Green function, we put $m_G(A, \cdot) := m_G(\chi_A, \cdot)$ and $m_G(a, \cdot) := m_G\{a, \cdot\}$. Obviously, $m_G(a, \cdot) = c^*_G(a, \cdot), a \in G$. More generally, $m_G(k\chi_{\{a\}}, \cdot) = [p^{(k)}_G(a, \cdot)]^k$. It is clear that $m_G(p, \cdot) \leq g_G(p, \cdot)$.

The above two generalizations lead us to the following definition. A family $d := (d_G)_G$ of functions $d_G : \mathbb{R}^+_n \times G \to \mathbb{R}^+$ is said to be a **generalized holomorphically contractible**
family if the following three conditions are satisfied:

(Normalization) \( \prod_{a \in D} [m_D(a, z)]^{p(a)} \leq d_D(p, z) \leq \inf_{a \in D} [m_D(a, z)]^{p(a)}, \ (p, z) \in \mathbb{R}_+^n \times \mathbb{D}; \)

(Contractibility) for any domains \( G \subset \mathbb{C}^n, \ D \subset \mathbb{C}^m, \ F \in \mathcal{O}(G, D), \) and \( q : D \to \mathbb{R}_+ \), we have \( d_D(q, F(z)) \leq d_G(q \circ F, z), \ z \in G; \)

(Monotonicity) for any domain \( D \) and \( p, q : G \to \mathbb{R}_+, \) if \( p \leq q, \) then \( d_G(q, \cdot) \leq d_G(p, \cdot). \)

If in the above definition one considers only integer-valued weights (like in the case of the generalized Möbius function), then we get the definition of a general holomorphically contractible family with integer-valued weights. As usually, we put \( d_G(A, \cdot) := d_G(\chi_A, \cdot) \) and \( d_G(a, \cdot) := d_G(\{a\}, \cdot). \)

One can prove that the generalized Green and Möbius functions are generalized holomorphically contractible families in the sense of the above definition. Moreover, one can prove that there exist minimal and maximal generalized holomorphically contractible families.

More details related to the above sketched theory may be found in our monograph [7] (Invariant Distances and Metrics in Complex Analysis, de Gruyter Expositions in Mathematics 9, Walter de Gruyter 1993) and recent survey article [8] (Invariant Distances and Metrics in Complex Analysis — revisited, Diss. Math. 430 (2005), 1–192).

REFERENCES

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Тази обзорна лекция е посветена на някои направления в теорията на инвариантните разстояния и метрики. В нея се дискутират основните инвариантни обекти и техните връзки с комплексния анализ.