SOME INTEGRAL GEOMETRIC RESULTS ON SETS OF CIRCLES IN THE SIMPLY ISOTROPIC SPACE

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The measurability with respect to the group of the general simply isotropic similitudes of sets of circles (elliptic and parabolic) is studied and also some formulas for the density are given.

1. Introduction. The simply isotropic space \( I_3^{(1)} \) is defined as a projective space \( \mathbb{P}_3(\mathbb{R}) \) with an absolute plane \( \omega \) and two complex conjugate straight lines \( f_1, f_2 \) in \( \omega \) with a (real) intersection point \( F \). All regular projectivities transforming the absolute figure into itself form the 8-parametric group \( G_8 \) of the general simply isotropic similitudes. Passing on to affine coordinates \((x, y, z)\) any similitude of \( G_8 \) can be written in the form

\[
\begin{align*}
\bar{x} &= c_1 + c_7 x \cos \varphi - y \sin \varphi, \\
\bar{y} &= c_2 + c_7 x \sin \varphi + y \cos \varphi, \\
\bar{z} &= c_3 + c_4 x + c_5 y + c_6 z,
\end{align*}
\]

where \( c_1, c_2, c_3, c_4, c_5, c_6 \neq 0, c_7 > 0 \) and \( \varphi \) are real parameters. We emphasize that more of the common material of the geometry of \( I_3^{(1)} \) can be found in [3].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [4], G. I. Drinfel’d [2], we study the measurability of sets of circles in \( I_3^{(1)} \) with respect to \( G_8 \).

2. Measurability of a set of circles of elliptic type. The conic \( k \) of one part in the nonisotropic plane \( \varepsilon \) is called a circle of elliptic type if the infinite points of \( k \) coincide with the intersection points of \( \varepsilon \) with \( f_1 \) and \( f_2 \) [3; p. 70]. We note the fact that any nonisotropic plane \( \varepsilon \), which is not a tangential plane of a sphere of parabolic type \( \Sigma \) and \( \varepsilon \cap \Sigma \neq \emptyset \), intersects \( \Sigma \) in a circle \( k \) of elliptic type [3; p. 72] (Fig. 1.a)).

Now, let \( k \) be a circle of elliptic type defined by the equations

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$$z = R(x^2 + y^2) + ax + by + c,$$

$$k : z = ux + vy + w,$$

where $R \neq 0$ and $(\alpha - u)^2 + (\beta - v)^2 - 4R(\gamma - w) > 0$. Under the action of (1) the circle $k(R, \alpha, \beta, \gamma, u, v, w)$ is transformed into the circle $k(R, \alpha, \beta, \gamma, u, v, w)$ according to

$$\overline{R} = c_6c_7^{-1} R,$$

$$\overline{\alpha} = c_7^{-1} [-2c_1c_6c_7^{-1} R + (c_6\alpha + c_4) \cos \varphi - (c_6\beta + c_5) \sin \varphi],$$

$$\overline{\beta} = c_7^{-1} [-2c_2c_6c_7^{-1} R + (c_6\alpha + c_4) \sin \varphi + (c_6\beta + c_5) \cos \varphi],$$

$$\overline{\gamma} = c_7^{-1} [(c_1^2 + c_2^2)c_6c_7^{-1} R + c_3c_7 + c_6c_7\gamma - (c_6\alpha + c_4)(c_1 \cos \varphi + c_2 \sin \varphi) +$$

$$+ (c_6\beta + c_5)(c_1 \sin \varphi - c_2 \cos \varphi)],$$

$$\overline{\alpha} = c_7^{-1} [(c_6u + c_4) \cos \varphi - (c_6v + c_5) \sin \varphi],$$

$$\overline{\beta} = c_7^{-1} [(c_6u + c_4) \sin \varphi + (c_6v + c_5) \cos \varphi],$$

$$\overline{\gamma} = c_7^{-1} [(c_3c_7 + c_6c_7w - (c_6u + c_4)(c_1 \cos \varphi + c_2 \sin \varphi) +$$

$$+ (c_6v + c_5)(c_1 \sin \varphi - c_2 \cos \varphi)].$$
The transformations (3) form the so-called associated group \( G_8 \) of \( G_8 \) [4; p. 34]. \( G_8 \) is isomorphic to \( G_8 \) and the invariant density under \( G_8 \) of the circles of elliptic type \( k \), if it exists, coincides with the invariant density under \( G_8 \) of the points \((R, \alpha, \beta, \gamma, u, v, w)\) in the set of parameters [4; p. 33]. The associated group \( G_8 \) has the infinitesimal operators

\[
Y_1 = 2R \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \gamma} + u \frac{\partial}{\partial w}, \quad Y_2 = 2R \frac{\partial}{\partial \beta} + \beta \frac{\partial}{\partial \gamma} + v \frac{\partial}{\partial w},
\]

\[
Y_3 = \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial w}, \quad Y_4 = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial u}, \quad Y_5 = \frac{\partial}{\partial \beta} + \frac{\partial}{\partial v},
\]

\[
Y_6 = R \frac{\partial}{\partial R} + \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} + \gamma \frac{\partial}{\partial \gamma} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w},
\]

\[
Y_7 = 2R \frac{\partial}{\partial R} + \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v},
\]

\[
Y_8 = -\beta \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \beta} - v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}.
\]

The integral invariant function \( f = f(R, \alpha, \beta, \gamma, u, v, w) \) satisfies the system of R. Deltheil [1; p. 28]

\[
Y_1(f) = 0, \quad Y_2(f) = 0, \quad Y_3(f) = 0, \quad Y_4(f) = 0, \quad Y_5(f) = 0,
\]

\[
Y_6(f) + 7f = 0, \quad Y_7(f) + 4f = 0, \quad Y_8(f) = 0
\]

and it has the form

\[
f = \frac{cR}{[(\alpha - u)^2 + (\beta - v)^2 - 4R(\gamma - w)]^4},
\]

where \( c = \text{const} \neq 0 \).

Thus, we can state:

**Theorem 1.** A set of circles of elliptic type (2) is measurable with respect to the group \( G_8 \) and has the invariant density

\[
(4) \quad dk = \frac{|R|dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw}{[(\alpha - u)^2 + (\beta - v)^2 - 4R(\gamma - w)]^4}.
\]

**Remark 1.** The orthogonal projection of the circle of elliptic type \( k \) on the coordinate plane \( Oxy \) is the Euclidean circle (Fig. 1.a)).

\[
(5) \quad \tilde{k} : R(x^2 + y^2) + (\alpha - u)x + (\beta - v)y + \gamma - w = 0, \quad z = 0
\]

with the center

\[
O\left(\frac{-\alpha - u}{2R}, \frac{-\beta - v}{2R}, 0\right)
\]

and the radius

\[
(6) \quad r = \frac{1}{2|R|} \sqrt{(\alpha - u)^2 + (\beta - v)^2 - 4R(\gamma - w)}.
\]

The quantity \( r \) is called the **radius of the circle of elliptic type** \( k \) [3; p. 70].
By (4) and (6) we have

\[ dk = \frac{1}{256r^8|R|^7} dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw. \tag{7} \]

**Remark 2.** The coordinate plane \( Oxy \) is a Euclidean plane and into \( Oxy \) the circles \( \tilde{k} \) defined by (5) have under the group \( \tilde{G}_4 \) of the similitudes

\[ \varphi = c_1 + c_7(x \cos \varphi - y \sin \varphi) \]
\[ \varphi = c_2 + c_7(x \sin \varphi + y \cos \varphi) \]

the invariant density \([4; p. 167]\)

\[ \tilde{d}k = \frac{dx_0 \wedge dy_0 \wedge d\gamma_0}{r^4}, \tag{8} \]

where

\[ x_0 = -\frac{\alpha - u}{2R}, \quad y_0 = -\frac{\beta - v}{2R}, \quad \gamma_0 = \frac{\gamma - w}{R}. \tag{9} \]

Differentiating (9), we have

\[ \frac{1}{2R} d\alpha = -dx_0 + \frac{\alpha - u}{2R^2} dR + \frac{1}{2R} du, \]
\[ \frac{1}{2R} d\beta = -dy_0 + \frac{\beta - v}{2R^2} dR + \frac{1}{2R} dv, \]
\[ \frac{1}{R} d\gamma = d\gamma_0 + \frac{\gamma - w}{R^2} dR + \frac{1}{R} dw. \tag{10} \]

By exterior product of (10) and \( dR \wedge du \wedge dv \wedge dw \) we obtain

\[ dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw = 4R^3 dR \wedge du \wedge dv \wedge dw \wedge dx_0 \wedge dy_0 \wedge d\gamma_0. \tag{11} \]

From (7), (8) and (11) it follows that

\[ dk = \frac{1}{64r^4R^3} dR \wedge du \wedge dv \wedge dw \wedge \tilde{d}k. \tag{12} \]

**Remark 3.** Denote by \( \psi \) the angle between both intersecting nonisotropic planes \( \varepsilon \) and \( Oxy \). Then we have \([3; p. 17]\)

\[ \psi = +\sqrt{u^2 + v^2} \]

and, therefore,

\[ \psi d\psi = udu + vdv. \]

From here it follows that

\[ du \wedge dv = \frac{\psi}{u} d\psi \wedge dv = \frac{\psi}{v} du \wedge d\psi \]

and the formula of the density (12) becomes

\[ dk = \frac{\psi}{64r^4R^4|u|} dR \wedge d\psi \wedge dv \wedge dw \wedge \tilde{k}, \tag{13} \]

\[ dk = \frac{\psi}{64r^4R^4|v|} dR \wedge du \wedge dv \wedge dw \wedge \tilde{k}, \]

respectively.

We summarize the foregoing results in the following
Theorem 2. The density for the circles of the elliptic type (4) satisfies the relations (7), (12) and (13).

3. Measurability of sets of circles of parabolic type. The conic \( k \) in the isotropic plane \( \iota \) is called a circle of the parabolic type if \( k \) touches the absolute plane \( \omega \) at the absolute point \( F \) [3; p. 70]. Any isotropic plane \( \iota \) intersects a sphere of parabolic type \( \Sigma \) in a circle \( k \) of parabolic type [3; p. 72] (Fig. 1.b).

Let \( k \) be a circle of parabolic type defined by the equations

\[
\begin{align*}
    k: \quad & z = R(x^2 + y^2) + \alpha x + \beta y + \gamma, \\
    & ux + vy + 1 = 0,
\end{align*}
\]

where \( R \neq 0 \), and \((u,v) \neq (0,0)\). Now, the corresponding associated group \( \mathcal{G}_8 \) has the infinitesimal operators

\[
\begin{align*}
    Y_1 &= -2R \frac{\partial}{\partial \alpha} - \alpha \frac{\partial}{\partial \gamma} + u^2 \frac{\partial}{\partial u} + uv \frac{\partial}{\partial v}, \\
    Y_2 &= -2R \frac{\partial}{\partial \beta} - \beta \frac{\partial}{\partial \gamma} + uv \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}, \\
    Y_3 &= \frac{\partial}{\partial \gamma}, \\
    Y_4 &= \frac{\partial}{\partial \alpha}, \\
    Y_5 &= \frac{\partial}{\partial \beta}, \\
    Y_6 &= R \frac{\partial}{\partial R} + \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} + \gamma \frac{\partial}{\partial \gamma}, \\
    Y_7 &= 2R \frac{\partial}{\partial R} + \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} + v \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \\
    Y_8 &= -\beta \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \beta} - v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}.
\end{align*}
\]

It is easy to verify that the system of R. Deltheil

\[
\begin{align*}
    Y_1(f) + 3uf &= 0, \\
    Y_2(f) + 3vf &= 0, \\
    Y_3(f) &= 0, \\
    Y_4(f) &= 0, \\
    Y_5(f) &= 0, \\
    Y_6(f) + 4f &= 0, \\
    Y_7(f) + 6f &= 0, \\
    Y_8(f) &= 0
\end{align*}
\]

has the solution

\[
f(R, \alpha, \beta, \gamma, u, v) = 0.
\]

From here it follows

Theorem 3. Sets of circles of parabolic type are not measurable with respect to the group \( \mathcal{G}_8 \).

REFERENCES


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ИНТЕГРАЛНО-ГЕОМЕТРИЧНИ РЕЗУЛТАТИ ЗА МНОЖЕСТВА ОТ СФЕРИ В ПРОСТО ИЗОТРОПНО ПРОСЪРТANСТВО

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Изследвана е измеримостта на множества сфери (еллиптични и параболични) относно групата на общите просто-изотропни подобности и са получени формулни за съответната гъстота.