МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006

BEST APPROXIMATIONS FOR THE BESSEL-LAGUERRE TYPE WEIERSTRASS TRANSFORM ON THE QUARTER PLANE*

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We give approximate real inversion formulas for the Weierstrass transform, associated with a system of partial differential operators on the region $\mathbb{K} = [0, +\infty[\times[0, +\infty[$, by using the best approximations and the theory of reproducing kernels.

1. Introduction. In this paper we consider a system of partial differential operators D_1 and D_2 defined on $\mathbb{K} = [0, +\infty[\times[0, +\infty[$, by

$$D_1 := \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}, \quad \alpha \geq 0, \quad t \geq 0,$$

$$D_2 := \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x}\frac{\partial}{\partial x} + x^2 D_1, \quad x > 0.$$

The eigenfunctions of this system are related to the Bessel and Laguerre functions and they satisfy a product formula which permits to develop a harmonic analysis associated to these operators.

In this paper we consider the Laguerre-type Weierstrass transform L_r associated with D_1 and D_2. This transform which generalizes the standard Weierstrass transform (see [4, 5]) solves the generalized heat equation:

$$\Delta u[(x,t), r] := (D_1 - D_2^2)u[(x,t), r] = \frac{\partial}{\partial r}u[(x,t), r] \tag{1}$$

on $\mathbb{K}\times]0, \infty[$ with the initial condition $u[(x,t), 0] = f(x,t)$ on $\mathbb{K}$.

We construct a family of Hilbert spaces and we exhibit explicitly their reproducing kernels. After that we prove the existence of the extremal function and we establish its estimate.

In the classical case [4, 5], the authors obtained analogous results by using the theory of reproducing kernels from the ideas of best approximations. Also the authors illustrated their numerical experiments by using computers.

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*Supported by UR04/15-02.
Key words: Weierstrass transform, reproducing kernel, best approximation.
2000 Mathematics Subject Classification: 44A15, 46E22, 35K05.

Throughout this paper we use the classic notation:

If (X, Ω) is a measurable space and m a positive measure on X, then $L^p(X) = L^p(X, m)$, $1 \leq p < +\infty$ represents the space of measurable functions $f : X \to \mathbb{C}$, such that

$$\|f\|_{p,m} = \left(\int_X |f(x)|^p dm(x) \right)^{1/p} < \infty.$$

2. The reproducing kernels. We begin this section by recalling some results about harmonic analysis associated with the differential operators D_1 and D_2. Next we exhibit the reproducing kernels of some Hilbert spaces associated to these operators.

The unique solution of the system

$$\begin{cases} D_1 u &= -\lambda^2 u \\ D_2 u &= -4\lambda \left(m + \dfrac{\alpha+1}{2} \right) u \\ u(0,0) &= 1, \ \dfrac{\partial u}{\partial x}(0,0) = \dfrac{\partial u}{\partial t}(0,0) = 0. \end{cases}$$

is the function $\varphi_{\lambda,m}$, $(\lambda, m) \in \Gamma = [0, \infty[\times \mathbb{N}$, defined by

$$\varphi_{\lambda,m}(x,t) = j_{\alpha-1/2}(\lambda t) \mathcal{L}_m^\alpha(\lambda x^2), \quad (x,t) \in \mathbb{K},$$

where j_α is the spherical Bessel function and $\mathcal{L}_m^\alpha, m \in \mathbb{N}$, is the Laguerre function defined on $[0, \infty[$ by

$$\mathcal{L}_m^\alpha(x) := e^{-x/2} \frac{L_m^{(\alpha)}(x)}{L_m^{(\alpha)}(0)},$$

$L_m^{(\alpha)}$ being the Laguerre polynomial of degree m and order α.

The harmonic analysis associated with these operators (Translation operators $\tau_{(x,t)}$, $(x,t) \in \mathbb{K}$, Convolution product $\star$, Bessel-Laguerre Fourier transform $\mathcal{F}, \dots$) has been developed by E. Jebbari and M. Sifi in [1] and [2].

Notations. We denote by:

- m_α : the measure defined on $\mathbb{K}$ by

$$dm_\alpha(x,t) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+1/2)} x^{2\alpha+1} t^{2\alpha} dx dt.$$

- γ_α : the measure on Γ, given by

$$d\gamma_\alpha(\lambda, m) = \frac{\lambda^{3\alpha+1}}{2^{2\alpha-1}\Gamma(\alpha+1/2)} L_m^{(\alpha)}(0) d\lambda \otimes \delta_m,$$

δ_m is the Dirac measure at m and $d\lambda$ is the Lebesgue measure.

- $\mathcal{W}(\mathbb{K}) = \{f \in L_\alpha^1(\mathbb{K}) \ / \mathcal{F}(f) \in L_\alpha^1(\Gamma)\}$.
- $H_\alpha^\nu(\mathbb{K})$, $\nu \in \mathbb{R}$, the space

$$H_\alpha^\nu(\mathbb{K}) := \left\{ f \in L_\alpha^2(\mathbb{K}) \ / \ [1 + \lambda^2(1+m^2)]^{\nu/2} \mathcal{F}(f) \in L_\alpha^2(\Gamma) \right\}.$$

The space $H^\nu_\alpha(\mathbb{K})$, provided with the inner product

$$\langle f,g\rangle_{H^\nu_\alpha} := \int_\Gamma [1+\lambda^2(1+m^2)]^\nu \mathcal{F}_L(f)(\lambda,m)\overline{\mathcal{F}_L(g)(\lambda,m)}d\gamma_\alpha(\lambda,m) \tag{2}$$

and the norm $\|f\|^2_{H^\nu_\alpha} = \langle f,f\rangle_{H^\nu_\alpha}$, is a Hilbert space.

Remark. In the rest of this paper, we suppose that $\nu > \dfrac{3}{2}\alpha+1$.

Proposition 1. *The Hilbert space $H^\nu_\alpha(\mathbb{K})$ admits the reproducing kernel:*

$$\mathcal{K}_\alpha[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}d\gamma_\alpha(\lambda,m),$$

that is:
i) For every $(y,s)\in\mathbb{K}$, the function $(x,t)\to\mathcal{K}_\alpha[(x,t),(y,s)]\in H^\nu_\alpha(\mathbb{K})$.
ii) For every $f\in H^\nu_\alpha(\mathbb{K})$ and $(y,s)\in\mathbb{K}$, we have

$$\langle f,\mathcal{K}_\alpha[\cdot,(y,s)]\rangle_{H^\nu_\alpha} = f(y,s).$$

Proof. *i*) Let $(y,s)\in K$. Since the function

$$(\lambda,m)\to\frac{\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}$$

belongs to $L^2_\alpha(\Gamma)$, from [1, Theorem 2.5] it follows that there exists a function in $L^2_\alpha(\mathbb{K})$, which we denote by $\mathcal{K}_\alpha[\cdot,(y,s)]$, such that

$$\mathcal{F}\Big(\mathcal{K}_\alpha[\cdot,(y,s)]\Big)(\lambda,m) = \frac{\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}. \tag{3}$$

Let $\Gamma_N := [0,N]\times\{0,1,\ldots,N\}$. Then we have

$$\mathcal{K}_\alpha[\cdot,(y,s)] = \lim_{N\to\infty}\int_{\Gamma_N}\frac{\varphi_{\lambda,m}(\cdot)\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}d\gamma_\alpha(\lambda,m),$$

in the $L^2_\alpha(\mathbb{K})$ sense.

So that there exists a subsequence $(N_p)_{p\in\mathbb{N}}$, such that

$$\mathcal{K}_\alpha[(x,t),(y,s)] = \lim_{p\to\infty}\int_{\Gamma_{N_p}}\frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}d\gamma_\alpha(\lambda,m), \quad \text{a.e } (x,t)\in\mathbb{K}.$$

Let

$$g_{N_p}(\lambda,m) := \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{(1+\lambda^2(1+m^2))^\nu}\mathbf{1}_{\Gamma_{N_p}}(\lambda,m), \quad (\lambda,m)\in\Gamma.$$

Since

$$\lim_{p\to\infty} g_{N_p}(\lambda,m) = \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu}\mathbf{1}_\Gamma(\lambda,m),$$

and

$$\sup_{(x,t)\in\mathbb{K}} \varphi_{\lambda,m}(x,t) = 1, \quad (\lambda,m)\in\mathbb{K}, \tag{4}$$

$$|g_{N_p}(\lambda,m)| \le \frac{1}{[1+\lambda^2(1+m^2)]^\nu}.$$

Then by the dominated convergence theorem $\mathcal{K}_\alpha[(x,t),(y,s)]$ is given by

$$\mathcal{K}_\alpha[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{[1+\lambda^2(1+m^2)]^\nu} d\gamma_\alpha(\lambda,m).$$

ii) Let $f \in \mathcal{W}(\mathbb{K}) \cap H^\nu_\alpha(\mathbb{K})$ and $(y,s)\in\mathbb{K}$. From (3) and (4) and [1, Theorem 2.5], we have

$$\langle f, \mathcal{K}_\alpha[\cdot,(y,s)]\rangle_{H^\nu_\alpha} = \int_\Gamma \mathcal{F}(f)(\lambda,m)\varphi_{\lambda,m}(y,s)d\gamma_\alpha(\lambda,m) = f(y,s).$$

The assertion *ii)* follows from the density of $\mathcal{W}(\mathbb{K})$ in $L^2_\alpha(\mathbb{K})$. □

Definition 1. *Let $r>0$. We define*

$$\mathcal{E}_r(x,t) := \int_\Gamma \exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m).$$

The generalized heat kernel E_r is given by

$$E_r[(x,t),(y,s)] := \tau_{(x,t)}\mathcal{E}_r(y,s); \quad (x,t),(y,s)\in\mathbb{K}.$$

Proposition 2. *Let $(x,t),\ (y,s)\in\mathbb{K}$ and $r>0$. Then, we have:*
i) The function $\mathcal{E}_r$ solves the generalized heat equation:

$$\Delta\mathcal{E}_r = \frac{\partial}{\partial r}\mathcal{E}_r,$$

where Δ is the operator given by (1).

ii) $\mathcal{F}\Big(E_r[(x,t),\cdot]\Big)(\lambda,m) = \exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{\lambda,m}(x,t).$

iii) $\int_\Gamma E_r[(x,t),(y,s)]dm_\alpha(x,t) = 1.$

iv) For fixed $(y,s)\in\mathbb{K}$, the function $u[(x,t),r] := E_r[(x,t),(y,s)]$ solves the generalized heat equation:

$$\Delta u[(x,t),r] = \frac{\partial}{\partial r}u[(x,t),r].$$

Proof. The assertion *i)* follows from Definition 1 and (3) by applying derivation under the integral sign.

The parts *ii)*, *iii)* and *iv)* can be easily proved. □

Definition 2. *The Bessel-Laguerre type Weierstrass transform is the integral operator given for $f \in L^2_\alpha(\mathbb{K})$ by*

$$L_r f(x,t) := \mathcal{E}_r * f(x,t) = \int_{\mathbb{K}} E_r[(x,t),(y,s)] f(y,-s) dm_\alpha(y,s).$$

Proposition 3. *i) The integral transform L_r, $r > 0$, solves the generalized heat equation:*

$$\Delta_L u[(x,t),r] = \frac{\partial}{\partial r} u[(x,t),r],$$

on $\mathbb{K}\times]0,\infty[$ *with the initial condition* $u[(x,t),0] = f(x,t)$ *on* $\mathbb{K}$.

ii) The integral transform L_r, $r > 0$, is a bounded linear operator from $H^\nu_\alpha(\mathbb{K})$ into $L^2_\alpha(\mathbb{K})$, and we have

$$\|L_r f\|_{2,m_\alpha} \leq c_\alpha(r) \|f\|_{H^\nu_\alpha},$$

where

$$c_\alpha(r) := \int_\Gamma \exp(-r\lambda^2[1 + 4(2m+\alpha+1)^2]) d\gamma_\alpha(\lambda, m).$$

Proof. *i)* This assertion follows from Definition 2 and Proposition 2, *iv*).

ii) Let $f \in H^\nu_\alpha(\mathbb{K})$. Applying Hölder's inequality, we get

$$\|L_r f\|_{2,m_\alpha} \leq \|E_r[(x,t),\cdot]\|_{\infty,m_\alpha} \|f\|_{2,m_\alpha}.$$

From [1, Theorem 2.5] and (4), we obtain

$$\|E_r[(x,t),\cdot]\|_{\infty,m_\alpha} \leq \int_\Gamma \exp(-r\lambda^2[1 + 4(2m+\alpha+1)^2]) d\gamma_\alpha(\lambda,m) := c_\alpha(r).$$

On the other hand, from [1, Theorem 2.5] we see that $\|f\|_{2,m_\alpha} \leq \|f\|_{H^\nu_\alpha}$. This proves ii). □

Definition 3. *Let $\mu > 0$. We define the Hilbert space $H_\mu(\mathbb{K}) = H_{\mu,\nu}(\mathbb{K})$ with the norm square:*

$$\|f\|^2_{H_\mu} := \mu \|f\|^2_{H^\nu_\alpha} + \|L_r f\|^2_{2,m_\alpha}.$$

As in Proposition 1, we obtain:

Proposition 4. *The Hilbert space $H_\mu(\mathbb{K})$ admits the following reproducing kernel:*

$$K_\mu[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s) d\gamma_\alpha(\lambda,m)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])}.$$

3. Extremal function for Bessel-Laguerre type Weierstrass transform. We can now state the main result of this paper.

Theorem 1. *For any $g \in L^2_\alpha(\mathbb{K})$ and for any $\mu > 0$, the infimum*

$$\inf_{f\in H^\nu_\alpha}\Big\{\mu\|f\|^2_{H^\nu_\alpha} + \|g - L_r f\|^2_{2,m_\alpha}\Big\} \tag{5}$$

is attained by a unique function $f^*_{\mu,g} = f^*_{\mu,\nu,g}$ and we have

$$f^*_{\mu,g}(x,t) = \int_{\mathbb{K}} g(y,s)Q_\mu[(x,t),(y,s)]dm_\alpha(y,s), \tag{6}$$

(*where* $Q_\mu[(x,t),(y,s)] = Q_{\mu,\nu}[(x,t),(y,s)]$)

$$= \int_\Gamma \frac{\exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])}d\gamma_\alpha(\lambda,m).$$

Proof. By Proposition 4 and [4, Theorem 2.1], the infimum given by (5) is attained by a unique function $f^*_{\mu,g}$, and the extremal function $f^*_{\mu,g}$ is represented by

$$f^*_{\mu,g}(y,s) = \langle g, L_r(K_\mu[\cdot,(y,s)])\rangle_{2,m_\alpha}, \quad (y,s)\in\mathbb{K},$$

where K_μ is the kernel given by Proposition 4.

Since for $(x,t)\in\mathbb{K}$,

$$L_r f(x,t) = \int_\Gamma \exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m),$$

we obtain

$$\begin{aligned} &L_r\Big(K_\mu[\cdot,(y,s)]\Big)(x,t)\\ &= \int_\Gamma \frac{\exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])}d\gamma_\alpha(\lambda,m)\\ &= Q_\mu[(x,t),(y,s)]. \end{aligned}$$

This gives (6). □

Corollary 1. *The extremal function $f^*_{\mu,g}$ in (6) can be estimated as follows:*

$$\|f^*_{\mu,g}\|^2_{2,m_\alpha} \le \frac{M_\alpha}{4\mu N_\alpha}\int_{\mathbb{K}} e^{-(y^2+s^2)}|g(y,s)|^2 dm_\alpha(y,s),$$

where

$$M_\alpha = \int_{\mathbb{K}} e^{-(y^2+s^2)}dm_\alpha(y,s) \quad \text{and} \quad N_\alpha = \Big(\int_\Gamma \frac{d\gamma_\alpha(\lambda,m)}{[1+\lambda^2(1+m^2)]^\nu}\Big)^{-1}.$$

Proof. Applying Hölder's inequality to relation (6), we obtain

$$|f^*_{\mu,g}(x,t)|^2 \le M_\alpha \int_{\mathbb{K}} e^{-(y^2+s^2)}|g(y,s)|^2|Q_\mu[(x,t),(y,s)]|^2 dm_\alpha(y,s).$$

From Fubini-Tonnelli's theorem we get

$$\|f^*_{\mu,g}\|^2_{2,m_\alpha} \le M_\alpha \int_\Omega e^{-(y^2+s^2)}|g(y,s)|^2\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha}dm_\alpha(y,s). \tag{7}$$

On the other hand, from [1, Theorem 2.5], we have

$$\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha} = \int_\Gamma |\mathcal{F}\Big(Q_\mu[\cdot,(y,s)]\Big)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

But for $(\lambda, m) \in \Gamma$ we have

$$\mathcal{F}_L\Big(Q_\mu[\cdot,(y,s)]\Big)(\lambda,m) = \frac{\exp(r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{-\lambda,m}(y,s)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp(2r\lambda^2[1+4(2m+\alpha+1)^2])}.$$

Then the inequality $(x+y)^2 \ge 4xy$ yields

$$\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha} \le \frac{1}{4\mu}\int_\Gamma \frac{1}{[1+\lambda^2(1+m^2)]^\nu} d\gamma_\alpha(\lambda,m).$$

From this inequality and (7) we deduce the result. □

Corollary 2. *Let $\delta > 0$ and $g, g_\delta \in L^2_\alpha(\mathbb{K})$ be such that*

$$\|g - g_\delta\|_{2,m_\alpha} \le \delta.$$

Then,

$$\|f^*_{\mu,g} - f^*_{\mu,g_\delta}\|_{H^\nu_\alpha} \le \frac{\delta}{2\sqrt{\mu}}.$$

Proof. From (6) and Fubini's theorem, we have for $(\lambda, m) \in \Gamma$,

$$\mathcal{F}_L(f^*_{\mu,g})(\lambda,m) = \frac{\exp(r\lambda^2[1+4(2m+\alpha+1])\mathcal{F}(g)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp(2r\lambda^2[1+4(2m+\alpha+1)^2])}. \tag{8}$$

Hence,

$$\mathcal{F}(f^*_{\mu,g} - f^*_{\mu,g_\delta})(\lambda,m) = \frac{\exp(r\lambda^2[1+4(2m+\alpha+1])\mathcal{F}_L(g-g_\delta)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp(2r\lambda^2[1+4(2m+\alpha+1)^2])}.$$

Using the inequality $(x+y)^2 \ge 4xy$, we obtain

$$[1+\lambda^2(1+m^2)]^\nu\Big|\mathcal{F}_L(f^*_{\mu,g} - f^*_{\mu,g_\delta})(\lambda,m)\Big|^2 \le \frac{1}{4\mu}|\mathcal{F}_L(g-g_\delta)(\lambda,m)|^2.$$

Thus we obtain

$$\|f^*_{\mu,g} - f^*_{\mu,g_\delta}\|^2_{H^\nu_\alpha} \le \frac{1}{4\mu}\|\mathcal{F}_L(g-g_\delta)\|^2_{2,\gamma_\alpha} \le \frac{1}{4\mu}\|g-g_\delta\|^2_{2,m_\alpha},$$

which gives the desired result. $\square$

Corollary 3. *Let $f \in H^{\nu}_{\alpha}(\mathbb{K})$ and $g = L_r f$. Then*

$$\|f^*_{\mu,g} - f\|^2_{H^{\nu}_{\alpha}} \to 0 \quad \textit{as} \;\; \mu \to 0.$$

Proof. From (8) we have

$$\mathcal{F}(f)(\lambda, m) = \exp(r\lambda^2[1 + 4(2m + \alpha + 1)^2])\mathcal{F}(g)(\lambda, m)$$

and

$$\mathcal{F}\Big(f^*_{\mu,g}\Big)(\lambda, m) = \frac{\exp(r\lambda^2[1 + 4(2m + \alpha + 1)^2])\mathcal{F}(g)(\lambda, m)}{1 + \mu[1 + \lambda^2(1 + m^2)]^{\nu} \exp(2r\lambda^2[1 + 4(2m + \alpha + 1)^2])}.$$

Thus,

$$\mathcal{F}(f^*_{\mu,g} - f)(\lambda, m) = -\frac{\mu[1 + \lambda^2(1 + m^2)]^{\nu} \exp(2r\lambda^2[1 + 4(2m + \alpha + 1)^2])\mathcal{F}(g)(\lambda, m)}{1 + \mu[1 + \lambda^2(1 + m^2)]^{\nu} \exp(2r\lambda^2[1 + 4(2m + \alpha + 1)^2])}.$$

Then we obtain

$$\|f^*_{\mu,g} - f\|^2_{H^{\nu}_{\alpha}} = \int_{\Gamma} h_{\mu,r,\nu}(\lambda, m)|\mathcal{F}(g)(\lambda, m)|^2 d\gamma_{\alpha}(\lambda),$$

with

$$h_{\mu,r,\nu}(\lambda, m) = \frac{\mu^2[1 + \lambda^2(1 + m^2)]^{3\nu} \exp(4r\lambda^2[1 + 4(2m + \alpha + 1)^2])}{(1 + \mu[1 + \lambda^2(1 + m^2)]^{\nu})^2 \exp(4r\lambda^2[1 + 4(2m + \alpha + 1)^2])}.$$

Since

$$\lim_{\mu \to 0} \; h_{\mu,r,\nu}(\lambda, m) = 0$$

and

$$|h_{\mu,r,\nu}(\lambda, m)| \le [1 + \lambda^2(1 + m^2)]^{\nu},$$

we obtain the result from the dominated convergence theorem. $\square$

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НАЙ-ДОБРИ АПРОКСИМАЦИИ ЗА ТРАНСФОРМАЦИЯТА НА ВАЙЕРЩРАС ОТ ТИПА НА БЕСЕЛ-ЛАГЕР ВЪРХУ КВАДРАНТ

Мохамед Сифи

В тази работа предлагаме приближени реални формули за обръщане на трансформацията на Вайерщрас, свързана със система частни диференциални уравнения в областта $\mathbb{K} = [0, +\infty[\times[0, +\infty[$. Като апарат се използуват метода на най-добрите апроксимации и теорията на репродуктивните ядра.

Ключови думи: трансформация на Вайерщрас, репродуктивни ядра, най-добри апроксимации.

2000 Mathematics Subject Classification: 44A15, 46E22, 35K05