IMPROVEMENT OF GRAPH THEORY WEI’S INEQUALITY

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Wei in [8] and [9] discovered a bound on the clique number of a given graph in terms of its degree sequence. In this note we give an improvement of this result.

We consider only finite non-oriented graphs without loops and multiple edges. A set of \( p \) vertices of a graph is called a \( p \)-clique if each two of them are adjacent. The greatest positive integer \( p \) for which \( G \) has a \( p \)-clique is called clique number of \( G \) and is denoted by \( \text{cl}(G) \). A set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The independence number \( \alpha(G) \) of a graph \( G \) is the cardinality of a largest independent set of \( G \).

In this note we shall use the following notations:

- \( V(G) \) is the vertex set of graph \( G \);
- \( N(v), v \in V(G) \) is the set of all vertices of \( G \) adjacent to \( v \);
- \( N(V), V \subseteq V(G) \) is the set \( \bigcap_{v \in V} N(v) \);
- \( d(v), v \in V(G) \) is the degree of the vertex \( v \), i.e. \( d(v) = |N(v)| \).

Let \( G \) be a graph, \( |V(G)| = n \) and \( V \subseteq V(G) \). We define

\[
W(V) = \sum_{v \in V} \frac{1}{n - d(v)}; \\
W(G) = W(V(G)).
\]

Wei in [8] and [9] discovered the inequality

\[
\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + d(v)}.
\]

Applying this inequality to the complementary graph of \( G \) we see that it is equivalent to the following inequality

\[
\text{cl}(G) \geq \sum_{v \in V(G)} \frac{1}{n - d(v)},
\]

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that is

\[ \text{cl}(G) \geq W(G). \]

Alon and Spencer [1] gave an elegant probabilistic proof of Wei’s inequality. In the present note we shall improve the inequality (1).

**Definition 1.** Let \( G \) be a graph, \(|V(G)| = n\) and \( V \subseteq V(G) \). The set \( V \) is called a \( \delta \)-set in \( G \), if

\[ d(v) \leq n - |V| \]

for all \( v \in V \).

**Example 1.** Any independent set \( V \) of vertices of a graph \( G \) is a \( \delta \)-set in \( G \) since \( N(v) \subseteq V(G) \setminus V \) for all \( v \in V \).

**Example 2.** Let \( V \subseteq V(G) \) and \(|V| \geq \max\{d(v), v \in V(G)\} \). Since \( d(v) \leq |V| \) for all \( v \in V(G), V(G) \setminus V \), is a \( \delta \)-set in \( G \).

The next statement obviously follows from Definition 1.

**Proposition 1.** Let \( V \) be a \( \delta \)-set in a graph \( G \). Then \( W(V) \leq 1 \).

**Definition 2.** A graph \( G \) is called an \( r \)-partite graph if

\[ V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j, \]

where the sets \( V_i, i = 1, \ldots, r \), are independent. If the sets \( V_i, i = 1, \ldots, r \), are \( \delta \)-sets in \( G \), then \( G \) is called generalized \( r \)-partite graph. The smallest integer \( r \) such that \( G \) is a generalized \( r \)-partite graph is denoted by \( \varphi(G) \).

**Proposition 2.** \( \varphi(G) \geq W(G) \).

**Proof.** Let \( \varphi(G) = r \) and

\[ V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j, \]

where \( V_i, i = 1, \ldots, r \), are \( \delta \)-sets in \( G \). Since \( V_i \cap V_j = \emptyset, i \neq j \), we have

\[ W(G) = \sum_{i=1}^{r} W(V_i). \]

According to Proposition 1, \( W(V_i) \leq 1, i = 1, \ldots, r \). Thus \( W(G) \leq r = \varphi(G) \).

Below (see Theorem 1) we shall prove that \( \text{cl}(G) \geq \varphi(G) \). Thus (1) follows from Proposition 2.

**Definition 3** [2]. Let \( G \) be a graph and \( v_1, \ldots, v_r \in V(G) \). The sequence \( v_1, \ldots, v_r \) is called an \( \alpha \)-sequence in \( G \) if the following conditions are satisfied:

(i) \( d(v_1) = \max\{d(v) \mid v \in V(G)\} \);

(ii) \( v_i \in N(v_1, \ldots, v_{i-1}) \) and \( v_i \) has maximal degree in the graph \( G[N(v_1, \ldots, v_{i-1})] \), \( 2 \leq i \leq r \).
Every α-sequence \( v_1, \ldots, v_r \) in the graph \( G \) can be extended to an α-sequence \( v_1, \ldots, v_s, \ldots, v_r \) such that \( N(v_1, \ldots, v_{r-1}) \) is a δ-set in \( G \). Indeed, if the α-sequence \( v_1, \ldots, v_s, \ldots, v_r \) is such that it is not contained in a \((r + 1)\)-clique (i.e. \( v_1, \ldots, v_s, \ldots, v_r \) is a maximal α-sequence in the sense of inclusion) then \( N(v_1, \ldots, v_{r-1}) \) is an independent set and, therefore, a δ-set in \( G \). However, there are α-sequences \( v_1, \ldots, v_r \) such that \( N(v_1, \ldots, v_{r-1}) \) is a δ-set but it is not an independent set.

**Theorem 1.** Let \( G \) be a graph and \( v_1, \ldots, v_r, r \geq 2 \), be an α-sequence in \( G \) such that \( N(v_1, \ldots, v_{r-1}) \) is a δ-set in \( G \). Then

(a) \( \varphi(G) \leq r \leq \text{cl}(G) \);

(b) \( r \geq \text{W}(G) \).

**Proof.** According to Definition 3, \( v_1, \ldots, v_r \) is an r-clique and thus \( r \leq \text{cl}(G) \). Since \( N(v_1, \ldots, v_{r-1}) \) is a δ-set, the graph \( G \) is a generalized r-partite graph, \( [6] \). Hence \( r \geq \varphi(G) \). The inequality (b) follows from (a) and Proposition 2.

**Remark.** Theorem 1 (b) was proved in \( [7] \) in the special case when \( N(v_1, \ldots, v_{r-1}) \) is independent set in \( G \).

**Definition 4.** Let \( G \) be a graph and \( v_1, \ldots, v_r \in V(G) \). The sequence \( v_1, \ldots, v_r \) is called β-sequence in \( G \) if the following conditions are satisfied:

(i) \( d(v_1) = \max\{d(v) \mid v \in V(G)\} \);

(ii) \( v_i \in N(v_1, \ldots, v_{i-1}) \) and \( d(v_i) = \max\{d(v) \mid v \in N(v_1, \ldots, v_{r-1})\} \), \( 2 \leq i \leq r \).

**Theorem 2.** Let \( v_1, \ldots, v_r \) be a β-sequence in a graph \( G \) such that

\[
d(v_1) + \cdots + d(v_r) \leq (r-1)n,
\]

where \( n = |V(G)| \). Then \( r \geq \text{W}(G) \).

**Proof.** According to \( [5] \), it follows from \( d(v_1) + \cdots + d(v_r) \leq (r-1)n \), that \( G \) is a generalized r-partite graph. Hence \( r \geq \varphi(G) \) and Theorem 2 follows from Proposition 2.

**Corollary.** Let \( G \) be a graph, \( |V(G)| = n \) and \( v_1, \ldots, v_r \) be a β-sequence in \( G \) which is not contained in \((r + 1)\)-clique. Then \( r \geq \text{W}(G) \).

**Proof.** Since \( v_1, \ldots, v_r \) is not contained in \((r + 1)\)-clique it follows that \( d(v_1) + \cdots + d(v_r) \leq (r-1)n \), \( [3] \).

**Theorem 3.** Let \( G \) be a graph, \( |V(G)| = n \) and \( v_1, \ldots, v_r, r \geq 2 \), be a β-sequence in \( G \) such that \( N(v_1, \ldots, v_{r-1}) \) is a δ-set in \( G \). Then \( r \geq \text{W}(G) \).

**Proof.** Since \( N(v_1, \ldots, v_{r-1}) \) is a δ-set according to \( [6] \) there exists an r-partition

\[
V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,
\]

where \( V_i, i = 1, \ldots, r \), are δ-sets and \( v_i \in V_i \). Thus, we have

\[
d(v_i) \leq n - |V_i|, \quad i = 1, \ldots, r.
\]

Summing up these inequalities we obtain that \( d(v_1) + \cdots + d(v_r) \leq (r-1)n \). Therefore Theorem 3 follows from Theorem 2.
REFERENCES


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ПОДОБРЕНИЕ НА НЕРАВЕНСВОТО НА WEI ОТ ТЕОРИЯ НА ГРАФИТЕ

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