МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2007
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2007
Proceedings of the Thirty Sixth Spring Conference of
the Union of Bulgarian Mathematicians
St. Konstantin & Elena resort, Varna, April 2–6, 2007

ON HOMOCLINIC SOLUTIONS OF A FOURTH-ORDER ODE ARISING IN WATER WAVE MODELS*

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We study the existence and symmetry of homoclinic solutions of a fourth-order differential equation arising in the theory of water waves. Two existence results are proved using the shooting method and a boundary point lemma.

1. Introduction. In this paper we investigate the existence and symmetry of traveling wave solutions of fifth-order Korteweg-de Vries equation of the form

$$u_t + \gamma u_{xxxxx} + \beta u_{xxx} = (F(u, u_x, u_{xx}))_x, \tag{1}$$

which appear in the classical water wave problem with gravity and capillarity. In (1) subscripts denote partial differentiation,

$$F(u, u_x, u_{xx}) = \mu(2uu'' + (u')^2) + f(u),$$

$\beta, \mu \in \mathbb{R}$, $\gamma > 0$ and $f(u)$ is a second-order polynomial. Looking for traveling waves $u(x,t) = u(x - ct)$, we obtain after appropriate scaling an equation of the form

$$\gamma u^{iv} = u'' + \mu(2uu'' + (u'')^2) + f(u). \tag{2}$$

A tipical example is the ODE

$$\frac{2}{15}u^{iv} - bu'' + au + \frac{3}{2}u^2 + \mu(\frac{1}{2}(u')^2 + (uu')') = 0,$$

derived by Craig and Groves [CrG], which describes gravity water waves on a surface with finite depth (see also [ChG], [GMYK], [P]).

In this work we study the existence of homoclinic solutions of the equation

$$\gamma u^{iv} = u'' + \mu(2uu'' + (u')^2) + u - u^2, \tag{3}$$

i.e., classical solutions $u = u(x)$ of Eq. (3), defined on $\mathbb{R}$ which satisfy the condition

$$(u, u', u'', u''')(x) \to (1, 0, 0, 0) \quad \text{as} \quad x \to \pm\infty. \tag{4}$$

The problem is inspired by the paper of Peletier, Rotariu–Bruma and Troy [PBT] where homoclinic solutions are studied for the stationary extended Fisher–Kolmogorov equation

$$\gamma u^{iv} = u'' + f(u), \quad \gamma > 0,$$

*This work was partially sponsored by the National Research Fund in Bulgaria, under Grant VU-MI-02/05.
2000 Mathematics Subject Classification: 34 C 37
Key words: Shooting method, water waves, boundary point lemma

by the shooting method. It is mentioned in [PBT] that this method can be applied to equations of the form (2) with $f(u) = -u - u^2$. Note that, under the change $u(x) = 1 + v(x/\sqrt{1+2\mu})$, Eq. (3) becomes

$$\frac{\gamma}{(1+2\mu)^2} v^{iv} = v'' + \frac{\mu}{1+2\mu}(2vv'' + v'^2) - v - v^2$$

which is of the mentioned form.

Eq. (3) is invariant with respect to the change of $u(x)$ by $u(-x)$. Therefore, we are looking for even solutions on $\mathbb{R}$ and consider Eq. (3) on $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. In order to have $u \in C^4(\mathbb{R})$, we require that $u'(0) = u'''(0) = 0$.

Our main result concerning the existence of even homoclinic solutions of Eq.(3) is as follows:

Theorem 1. *Let $0 < \gamma \leq (1+2\mu)^2/4$ if $-1/2 < \mu \leq 1/2$ or $0 < \gamma \leq 2\mu$ if $\mu > 1/2$. Then, Eq. (3) admits an even homoclinic solution $u(x)$ such that $-1/2 < u(x) < 1$ for all $x \in \mathbb{R}$, $u(0) < 0$ and $u'(x) > 0$ for all $x > 0$.*

By our next result we give conditions on coefficients and u which ensure that a homoclinic solution of Eq.(3) is necessarily symmetric with respect to a point of minimum $y \in \mathbb{R}$, i.e.

(5) $$u(y+x) = u(y-x), \ \forall x \in \mathbb{R},$$

(6) $$u(x) \geq u(y), \ \forall x \in \mathbb{R}.$$

Theorem 2. *Let $\mu \in [0,1]$ and $\gamma > 0$ be such that $(1+2\mu)^2 \geq 4\gamma \geq 6\mu + 3\mu^2$ and let the solution of (3), (4) satisfy $-1/2 < u(x) < 1$ for all $x \in \mathbb{R}$. Then, there exists a point $y \in \mathbb{R}$ such that (5) and (6) hold. If $(1+2\mu)^2 \geq 4\gamma$ and $\mu \in (-1/2, 0)$, then the assertions (5) and (6) are still valid provided u has an unique local minimum point.*

2. Sketch of proofs of basic results. To prove Theorem 1 we use the shooting method conserning the initial value problem

$$(P): \begin{cases} \gamma u^{iv} = u'' + \mu(2uu'' + (u')^2) + u - u^2, \\ (u, u', u'', u''')(0) = (\alpha, 0, \beta, 0). \end{cases}$$

We require that $\beta \geq 0$ and seek for a solution of (P) which is increasing on $\mathbb{R}^+$. Let $f(s) = s - s^2$ and $F(s) = \int_s^1 f(t)\,dt = \frac{1}{6}(1-s)^2(1+2s)$. We have $F(s) \geq 0$ iff $s \geq -1/2$. Eq. (3) has a prime integral (conservation law). Indeed, if we multiply (3) by $2u'$, integrate over $(-\infty, x)$ and use (4), then we obtain

(7) $$2\gamma u'u''' - \gamma u''^2 - u'^2 - 2\mu u u'^2 + 2F(u) = 0,$$

which is known as the conservation law.

We choose $x = 0$ in (7) and α in the interval $I := (-1/2, 1)$ and obtain $\gamma\beta^2 = 2F(\alpha)$. So that $\beta = \beta(\alpha) = \sqrt{\frac{2}{\gamma}F(\alpha)}$.

Problem (P) has a unique local solution $u = u(x, \alpha)$. If $\alpha \in I$, then $\beta(\alpha) > 0$ and $u'(x, \alpha) > 0$ in a right neighborhood of 0. Then, the number

(8) $$\xi(\alpha) := \sup\{x > 0 : \ u'(t, \alpha) > 0, \ t \in (0, x)\}$$

is well defined for any $\alpha \in I$. Define as well as the "shooting set"

$$\mathcal{S} := \left\{ \widehat{\alpha} > -1/2 : \ 0 < \xi(\alpha) < \infty, \ u(\xi(\alpha), \alpha) < 1, \ \forall \alpha \in (-\frac{1}{2}, \widehat{\alpha}) \right\}. \tag{9}$$

Lemma 3. *If* $0 < \gamma \leq \frac{(1+2\mu)^2}{4}$, *then*
(a) $u'(\xi(\alpha), \alpha) = 0$ *for all* $\alpha \in \mathcal{S}$,
(b) $\xi \in C^1(\mathcal{S})$,
(c) $\mathcal{S}$ *is an open set.*

The proof follows exactly the same arguments as those of Lemma 2.2 in [PBT]. Next, we have

Lemma 4. *Let* $\alpha^* = \sup S$. *Then* $-1/2 < \alpha^* < 0$.

The proof is complicated. It needs the following technical results

Lemma 5. *Let* $u \in C^2([a,b])$ *and suppose that:* $u'(a) = 0, u(a) \geq 0, u''(x) \geq 0, x \in [a,b]$ *and* u'' *is a nondecreasing function. Then* $u'^2(x) \leq 2u(x)u''(x)$ *for all* $x \in [a,b]$.

We know three different proofs.

The shortest is based on the inequality $\int\limits_a^x (u''(x) - u''(t))\, u'(t)\, dt \geq 0, \quad x \in [a,b].$

Note that the reverse inequality $u'^2(x) \geq u(x)u''(x)$ is known as Laguere's inequality. It is satisfied for a class of polynomials.

Below we also need the Maximum principle and so called Boundary Point Lemma [PW, Theorem 4] which we formulate as:

Lemma 6. *Suppose that* $u \in C^2([a,b])$ *is a nonconstant solution of differential inequality* $u''(x) - c(x)u(x) \geq 0, \ x \in [a,b]$ *where* $c(x) \geq 0$ *for all* $x \in [a,b]$. *If* u *has a nonnegative maximum at* a, *then* $u'(a) < 0$. *If* u *has a nonnegative maximum at* b, *then* $u'(b) > 0$.

We assume $\mu \neq 0$ in the further considerations, because the case $\mu = 0$ is considered in [PBT]. The key step in the proof of Theorem 1 is

Lemma 7. *Let* $\mu > -\frac{1}{2}$ *and* $0 < \gamma \leq \frac{(1+2\mu)^2}{4}$ *if* $\mu \leq \frac{1}{2}$ *and* $0 < \gamma \leq 2\mu$ *if* $\mu > \frac{1}{2}$. *Then* $\xi(\alpha^*) = +\infty$ *and* $u(x, \alpha^*) \to 1$ *as* $x \to +\infty$.

The final part of the proof of Theorem 1 is to show that the solution $u(x) = u(x, \alpha^*)$, constructed in previous lemma satisfies as well $(u', u'', u''')(x) \to (0,0,0)$ as $x \to +\infty$.

Now, we outline the main steps in the proof of Theorem 2. Let u be a solution of (3), (4). The function $v = 1 - u$ satisfies the equation

$$\gamma v^{iv} - (1 + 2\mu - 2\mu v)v'' + v = v^2 - \mu v'^2. \tag{10}$$

Let v takes it maximum value at $y \in \mathbb{R}$. We may set $y = 0$ since (3) and (10) are autonomous, i.e. not depending on x. Define $v_1(x) = v(x)$ for $x > 0$, $v_2(x) = v(-x)$ for $x > 0$ and $z(x) = v_1(x) - v_2(x)$ for $x \geq 0$. Then, $(z, z', z'')(0) = (0,0,0)$. If $z'''(0) = 0$, then by the existence uniqueness theorem it follows that $z(x) \equiv 0$ for $x \geq 0$ which implies that v and u are symmetric on $\mathbb{R}$. Assume that $z'''(0) > 0$. Then, there exists $\delta > 0$ such that $z'(x) > 0$, $x \in (0, \delta)$ and let

$$x_1 = \sup\{x > 0 : z'(t) > 0, \ t \in (0, x)\}. \tag{11}$$

We have $x_1 < +\infty$ because $z(+\infty) = v(+\infty) - v(-\infty) = 0$. Eq. (10) is equivalent to the system

$$
\begin{cases} v'' - \mu_1 v &= w, \\ w'' - \mu_2 w &= \dfrac{1}{\gamma}(1 - \mu(\mu_{10} + \dfrac{1}{2}\mu_{20}))v^2 + \dfrac{\mu^2}{2\gamma^2}v^3, \end{cases} \tag{12}
$$

where

$$
\begin{aligned} &\mu_1 = \mu_{10} - \frac{\mu}{2\gamma}v, \;\; \mu_2 = \mu_{20} - \frac{\mu}{\gamma}v, \;\; \mu_{10} = \frac{1}{2\gamma}(1 + 2\mu - \sqrt{D}) \\ &\mu_{20} = \frac{1}{2\gamma}(1 + 2\mu + \sqrt{D}), \;\; D = (1 + 2\mu)^2 - 4\gamma. \end{aligned} \tag{13}
$$

Then, v_1 and v_2 satisfy (12) and let $w = w_1 - w_2$ where $w_j = v_j'' - \mu_1 v_j, \quad j = 1, 2$ and $h(v) = (1 - \mu(\mu_{10} + \frac{1}{2}\mu_{20}))v^2 + \frac{\mu^2}{2\gamma}v^3$. It follows that

$$
\begin{cases} z'' - (\mu_{10} - \dfrac{\mu}{2\gamma}(v_1 + v_2))z &= w, \\ w'' - (\mu_{20} - \dfrac{\mu}{\gamma}v_2)w &= \dfrac{1}{\gamma}(h(v_1) - h(v_2)) - \dfrac{\mu}{\gamma}w_1 z. \end{cases} \tag{14}
$$

Let $\mu \in [0, 1]$ We can apply Lemma 6 to the systems (12) and (14) provided

$$
\mu_{10} - \frac{\mu}{2\gamma}(v_1 + v_2) > 0, \; \mu_{20} - \frac{\mu}{\gamma}v_2 > 0, \; 1 - \mu(\mu_{10} + \frac{1}{2}\mu_{20}) \geq 0. \tag{15}
$$

Since $v_j, j = 1, 2$, takes its maximum at 0 and $v_1(0) = v_2(0) = 1 - \alpha$, $\alpha \in (-1/2, 1)$, the last conditions are satisfied if

$$
\mu_{10} - \frac{\mu}{\gamma}(1 - \alpha) > 0, \quad \mu_{20} - \frac{\mu}{\gamma}(1 - \alpha) > 0, \quad 1 - \mu(\mu_{10} + \frac{1}{2}\mu_{20}) \geq 0. \tag{16}
$$

The inequality $\mu_{10} - \frac{\mu}{\gamma}(1 - \alpha) > 0$ holds if $\frac{1}{2\gamma}(1 + 2\mu - \sqrt{D}) \geq \frac{3\mu}{2\gamma}$. The last inequality is equivalent to $4\gamma \geq 6\mu + 3\mu^2$ which is assumption of Theorem 2. The inequality $\mu_{20} - \frac{\mu}{\gamma}(1 - \alpha) > 0$ holds if $\frac{1}{2\gamma}(1 + 2\mu + \sqrt{D}) \geq \frac{3\mu}{2\gamma}$ which is equivalent to $1 - \mu + \sqrt{D} \geq 0$, which is fulfilled since $\mu \leq 1$. Finally, the inequality $1 - \mu(\mu_{10} + \frac{1}{2}\mu_{20}) \geq 0$ for $\mu > 0$ is equivalent to $\sqrt{D} \geq 0 \geq \dfrac{6\mu + 3\mu^2 - 4\gamma}{\mu}$ which is true by the assumption of Theorem 2. By $w(0) = 0$, $w(x_1) < 0$ and Lemma 6 we obtain that $w(x) < 0$, $0 < x \leq x_1$. By $z(0) = 0$ and $z(x_1) > 0$ again by Lemma 6, we have $z(x) > 0$, $0 < x \leq x_1$ and $z'(0) > 0$ which contradicts to $z'(0) = 0$. Then, $z(x) \equiv 0$ for $x \geq 0$. If $\mu \in (-1/2, 0)$, then we cannot apply Lemma 6 to system (14) because the term $-\frac{\mu}{\gamma}w_1 z$ is negative. We can avoid this difficulty assuming that $v(x) = 1 - u(x)$ has unique local maximum point and apply Lemma 6 twice to the equivalent system

$$
\begin{cases} v'' - \mu_1 v &= w, \\ w'' - \mu_2 w &= \dfrac{\mu}{\gamma}v'^2 + \dfrac{1 - \mu\mu_{20}}{\gamma}v^2, \end{cases}
$$

where $\mu_1 = \mu_{10} - \frac{\mu}{\gamma}v$, $\mu_2 = \mu_{20}$. Here μ_{10} and μ_{20} are defined in (13).

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ВЪРХУ ХОМОКЛИНИЧНИТЕ РЕШЕНИЯ НА ОБИКНОВЕНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ОТ ЧЕТВЪРТИ РЕД ОПИСВАЩИ ВОДНИ ВЪЛНИ

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В работата се изучават съществуването и симетрията на хомоклинични решения на диференциални уравнения от четвърти ред, които се срещат в теорията на водните вълни. Доказани са два резултата с използване на метод на стрелбата и лема за граничните точки.