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AFFINE APPROXIMATION OF RATIONAL POWER MATRIX FUNCTIONS*

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We give nonlocal rigorous estimates for the accuracy of affine approximation for matrix valued functions $X \mapsto X^{1/m}$, where $m \geq 2$ is a positive integer. The results are obtained by the technique of Lyapunov majorants and fixed point principles.

Introduction and notations. In this paper we derive nonlocal estimates for the accuracy of affine approximations for matrix valued functions $\mathbb{S}_+^{n\times n} \to \mathbb{S}_+^{n\times n}$, defined by $X \mapsto X^p$, where $p \in \mathbb{R}$ and $\mathbb{S}_+^{n\times n} \subset \mathbb{C}^{n\times n}$ is the set of Hermitian positive definite $n \times n$ matrices. Further on, the particular case $p = 1/m$, $2 \leq m \in \mathbb{N}$, is considered. We stress that matrix power functions and their perturbations are subject to intensive investigations due to their application in a number of applied problems, see e.g. [3].

We use the following notations: $A^{\mathrm{H}} := \overline{A}^{\top}$ – the complex conjugate transpose of the matrix A; $\mathrm{spect}(A) = \{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}$ – the full spectrum of the matrix $A \in \mathbb{K}^{n\times n}$, i.e. the collection of its eigenvalues $\lambda_k(A)$, counted according to their algebraic multiplicity; $\lambda_{\max}(A) \geq \lambda_{\min}(A) > 0$ – the maximum and minimum eigenvalues of $A \in \mathbb{S}_+^{n\times n}$; $\|\cdot\|_2$ – the Euclidean norm in $\mathbb{C}^n$ or the spectral norm in $\mathbb{C}^{n\times n}$; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius norm in $\mathbb{C}^{n\times n}$; $A \otimes B$ – the Kronecker product of the matrices A and B.

For a linear operator $\mathcal{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ we define its matrix $\mathrm{Mat}(\mathcal{L}) \in \mathbb{C}^{n^2\times n^2}$ from $\mathrm{vec}(\mathcal{L}(Y)) = \mathrm{Mat}(\mathcal{L})\mathrm{vec}(Y)$, $Y \in \mathbb{C}^{n\times n}$, where $\mathrm{vec}(Y) \in \mathbb{C}^{n^2}$ is the column-wise vector representation of Y. If, in particular, $\mathcal{L}(Y) = \sum_k A_k Y B_k$ then $\mathrm{Mat}(\mathcal{L}) = \sum_k B_k^{\top} \otimes A_k$.

For a matrix $A \in \mathbb{C}^{n\times n}$ and a number $p \in \mathbb{R}$ the definition of the power A^p is not trivial. We define the quantity A^p in the following two cases: *First*, when $A \in \mathbb{S}_+^{n\times n}$ we set $A^p := U\Lambda^p U^{\mathrm{H}}$, where $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_k = \lambda_k(A) > 0$ and $U \in \mathbb{C}^{n\times n}$ is the unitary matrix such that $U^{\mathrm{H}} A U = \Lambda$; *Second*, when A is not Hermitian positively definite but has positive pairwise distinct eigenvalues we define $A^p := U\Lambda^p U^{-1}$, where $U \in \mathbb{C}^{n\times n}$ is nonsingular. In both cases λ_k^p is the (real) positive p–th degree of $\lambda_k > 0$.

We consider the first case with $p = 1/m$, $2 \leq m \in \mathbb{N}$. The treatment of the second case is similar.

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Let the Hermitian matrix $E \in \mathbb{C}^{n\times n}$ be such that $A + E \in \mathbb{S}_+^{n\times n}$. This is fulfilled if $\|E\|_2 < \lambda_{\min}(A)$. Then, we have the representation

$$(1) \qquad (A+E)^{1/m} = A^{1/m} + F_m(A,E) + G_m(A,E),$$

where the linear operator $F_m(A,\cdot) := \mathcal{F}(1/m, A)(\cdot)$ is the Frechét derivative of the function $X \mapsto X^{1/m}$ at the point $X = A$, and the expression $G_m(A,E) := (A+E)^{1/m} - A^{1/m} - F_m(A,E)$ contains terms of second and higher order in E. We recall that the Frechét derivative $\mathcal{F}(p,A)$ of the function $X \mapsto X^p$ at the point $X = A \in \mathbb{S}_+^{n\times n}$ is a linear operator $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that $(A+E)^p = A^p + \mathcal{F}(p,A)(E) + \mathrm{O}(\|E\|^2)$, $E \to 0$.

The Frechét derivatives for rational matrix valued functions has been studied in [1] as a special type of Lyapunov operators. In particular, for $m \in \mathbb{N}$ one has

$$\mathcal{F}(m,A)(E) = \sum_{k=0}^{m-1} A^{m-1-k} E A^k, \ \mathcal{F}(1/m, A) = \mathcal{F}^{-1}(m, A^{1/m}).$$

For any $B, Y \in \mathbb{C}^{n\times n}$ denote

$$(2) \qquad H_m(B,Y) := (B+Y)^m - B^m - \mathcal{F}(m,B)(Y).$$

Thus, when Y is small we have $\|H_m(B,Y)\| = \mathrm{O}(\|Y\|^2)$.

Problem statement. The aim of this paper is to give rigorous bounds for the norm of $G_m(A,E)$ in (1) as a function of $\varepsilon := \|E\|_{\mathrm{F}}$ and, thus, estimating the accuracy of the affine approximation $(A+E)^{1/m} \simeq A^{1/m} + F_m(A,E)$ of the perturbed quantity $(A+E)^{1/m}$ for small E.

Taking the m–th degree of both sides of (1), then in view of (2) with $B := A^{1/m}$, we obtain

$$A + E = A + \mathcal{F}(m,B)(F_m) + \mathcal{F}(m,B)(G_m) + H_m(B, F_m + G_m).$$

Since $\mathcal{F}(m,B)(F_m) = \mathcal{F}(m,A^{1/m}) \circ \mathcal{F}(1/m,A)(E) = E$, we have $F_m = \mathcal{F}^{-1}(m, A^{1/m})(E)$ and $\|F_m\|_{\mathrm{F}} \le \varphi_m \varepsilon$, where

$$\varphi_m = \varphi_m(A) := \|\mathcal{F}^{-1}(m, A^{1/m})\|.$$

Here the norm $\|\mathcal{L}\|$ of the linear operator $\mathcal{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is defined by

$$\|\mathcal{L}\| := \max\{\|\mathcal{L}(Y)\|_{\mathrm{F}} : \|Y\|_{\mathrm{F}} = 1\} = \|\mathrm{Mat}(\mathcal{L})\|_2,$$

where $\mathrm{Mat}(\mathcal{L}) \in \mathbb{C}^{n^2\times n^2}$ is the matrix of $\mathcal{L}$.

An expression for φ_m may be derived using the results from [1], namely

$$\varphi_m = \frac{\lambda_{\min}^{1/m-1}(A)}{m}.$$

Furthermore, setting $Z := G_m$ we obtain the operator equation

$$(3) \qquad Z = \Pi(E,Z) := -\mathcal{F}^{-1}(m, A^{1/m})(H_m(A^{1/m}, F_m(A,E) + Z)).$$

Since $\Pi(0,0) = 0$, it follows that for small E the operator equation (3) has a small solution $Z = Z(E)$ vanishing together with E. To estimate the quantity $\|Z(E)\|_{\mathrm{F}}$ as a function of $\varepsilon = \|E\|_{\mathrm{F}}$ we use the technique of Lyapunov majorants.

Suppose that $\|Z\|_{\mathrm{F}} \le \rho$. Then, it follows from the definition of Π that there exists a polynomial m–th degree Lyapunov majorant $h_m(\varepsilon,\rho)$ for Π such that

$$\begin{aligned}\|\Pi(E,Z)\|_{\mathrm F} &\le \max\{\|\Pi(E,Z)\|_{\mathrm F} : \|E\|_{\mathrm F}\le\varepsilon, \|Z\|_{\mathrm F}\le\rho\} \le h_m(\varepsilon,\rho),\\ h_m(\varepsilon,\rho) &= \sum_{k=0}^{m} a_k(\varepsilon)\rho^k.\end{aligned}$$

For a given $\varepsilon>0$ consider the majorant equation

$$\rho := h_m(\varepsilon,\rho)$$

for determining $\rho=\rho(\varepsilon)>0$. The coefficients $a_k=a_k(\varepsilon)$ are non-decreasing functions of $\varepsilon\ge 0$ and satisfy $a_0(0)=a_1(0)=0$. At the same time the function $h_m(\varepsilon,\cdot)$ is convex. Hence, there exists a quantity $\varepsilon_m>0$ such that for $\varepsilon<\varepsilon_m$, the majorant equation has two nonnegative solutions (depending on ε). One of them is "small" and vanishes together with ε. For $\varepsilon=\varepsilon_m$ both solutions coincide. Finally, for $\varepsilon>\varepsilon_m$ the majorant equation has no positive solutions.

In the sequel we also need the (unique) root $\rho_m=\rho_m(\varepsilon)$ of the equation $1 = \partial h_m(\varepsilon,\rho)/\partial\rho$ in ρ. This root exists for all $\varepsilon\ge 0$ such that $a_1(\varepsilon)<1$.

Denote the small solution of the majorant equation by $f_m(\varepsilon)$. The function $f_m : [0,\varepsilon_m]\to\mathbb{R}_+$ is continuous, increasing and $f_m(0)=0$.

Thus, for $\varepsilon\le\varepsilon_m$ and $\|Z\|_{\mathrm F}\le f_m(\varepsilon)$ we have $\|\Pi(E,Z)\|_{\mathrm F}\le f_m(\varepsilon)$. Therefore, the operator $\Pi(E,\cdot)$ transforms the closed convex set $\mathbf{B}_\varepsilon := \{Z\in\mathbb{C}^{n\times n} : \|Z\|_{\mathrm F}\le f_m(\varepsilon)\}$ into itself. According to the Schauder fixed point principle, there exists $Z\in\mathbf{B}_\varepsilon$ such that $Z=\Pi(E,Z)$. In this way, recalling that we have set $Z=G_m$, we find that

$$\|G_m(A,E)\|_{\mathrm F}\le f_m(\varepsilon),\ \varepsilon\in[0,\varepsilon_m],\ \varepsilon=\|E\|_{\mathrm F}. \tag{4}$$

is the desired nonlocal rigorous bound for the norm of the higher order terms G_m in the representation (1).

Main results. In this section we give easily computable bounds $\widehat{f}_m(\varepsilon)\ge f_m(\varepsilon)$, $\varepsilon\le\widehat{\varepsilon}_m\le\varepsilon_m$, for the expressions $f_m(\varepsilon)$ in (4). In this way we obtain accuracy estimates for the affine approximations of the matrix functions $X\mapsto X^{1/m}$ at points $X=A\in\mathbb{S}_+^{n\times n}$ for $m\ge 2$.

We consider first the case $m=2$ when an explicit expression for $f_2(\varepsilon)$ may be found. For $m>2$ we use the technique, proposed in [2], in order to find a bound $\widehat{f}_m(\varepsilon)\ge f_m(\varepsilon)$.

(i) The case $m=2$**.** Here we have $H_2(A^{1/2},F_2+Z) = -(F_2+Z)^2$. Hence, the Lyapunov majorant is $h_2(\varepsilon,\rho)=a_0(\varepsilon)+a_1(\varepsilon)\rho+a_2\rho^2$ with $a_0(\varepsilon)=\varphi_2^3\varepsilon^2$, $a_1(\varepsilon)=2\varphi_2^2\varepsilon$ and $a_2=\varphi_2$. The majorant equation is $a_0(\varepsilon)-(1-a_1(\varepsilon))\rho+a_2\rho^2=0$ and, hence, ε_2 is the largest root of the equation $d(\varepsilon):=(1-a_1(\varepsilon))^2-4a_2a_0(\varepsilon)=0$, or $a_1(\varepsilon)+2\sqrt{a_2a_0(\varepsilon)}=1$. In this case the root $\varepsilon_2=1/(4\varphi_2^2)$ is unique. Hence

$$\begin{aligned}f_2(\varepsilon) &= \frac{1-a_1(\varepsilon)-\sqrt{(1-a_1(\varepsilon))^2-4a_2a_0(\varepsilon)}}{2a_2}\\ &= \frac{2a_0(\varepsilon)}{1-a_1(\varepsilon)+\sqrt{(1-a_1(\varepsilon))^2-4a_2a_0(\varepsilon)}},\ \varepsilon\in[0,\varepsilon_2],\end{aligned}$$

or

$$f_2(\varepsilon)=\frac{2\varphi_2^3\varepsilon^2}{1-2\varphi_2^2\varepsilon+\sqrt{1-4\varphi_2^2\varepsilon}},\ \varepsilon\in[0,\varepsilon_2],\ \varepsilon_2:=\frac{1}{4\varphi_2^2}. \tag{5}$$

(ii) The case $m = 3$. Here the majorant equation

(6) $$\rho = h_3(\varepsilon, \rho) = a_0(\varepsilon) + a_1(\varepsilon)\rho + a_2(\varepsilon)\rho^2 + a_3\rho^3$$

is cubic, where

$$\begin{aligned} a_0(\varepsilon) &= \varphi_3^3\varepsilon^2(3\alpha_3 + \varphi_3\varepsilon), \\ a_1(\varepsilon) &= 3\varphi_3^2\varepsilon(2\alpha_3 + \varphi_3\varepsilon), \\ a_2(\varepsilon) &= 3\varphi_3(\alpha_3 + \varphi_3\varepsilon), \\ a_3 &= \varphi_3 \end{aligned}$$

and

$$\alpha_m := \|A^{1/m}\|_2 = \lambda_{\max}^{1/m}(A).$$

Let, for a given $\varepsilon > 0$ such that $a_1(\varepsilon) < 1$, the quantity $\rho_3(\varepsilon)$ be the unique solution of the equation $1 = a_1(\varepsilon) + 2a_2(\varepsilon)\rho + 3a_3\rho^2$, i.e.

$$\rho_3(\varepsilon) = \frac{1 - a_1(\varepsilon)}{a_2(\varepsilon) + \sqrt{a_2^2(\varepsilon) + 3a_3(1 - a_1(\varepsilon))}}.$$

For the small solution of the majorant equation, it holds that $f_3(\varepsilon) \leq \rho_3(\varepsilon)$. Hence, $\widehat{h}_3(\varepsilon, \rho) := a_0(\varepsilon) + a_1(\varepsilon)\rho + \widehat{a}_2(\varepsilon)\rho^2$, with $\widehat{a}_2(\varepsilon) := a_2(\varepsilon) + a_3\rho_3(\varepsilon)$, is again a Lyapunov majorant in the form of a second degree polynomial in ρ such that $h_3(\varepsilon, \rho) \leq \widehat{h}_3(\varepsilon, \rho)$. So that we may apply the estimates already obtained for $m = 2$. Let $\widehat{\varepsilon}_m > 0$ be the largest root of the equation $1 = a_1(\varepsilon) + 2\sqrt{a_0(\varepsilon)\widehat{a}_2(\varepsilon)}$. Then,

(7) $$f_3(\varepsilon) \leq \widehat{f}_3(\varepsilon) := \frac{2a_0(\varepsilon)}{1 - a_1(\varepsilon) + \sqrt{(1 - a_1(\varepsilon))^2 - 4a_0(\varepsilon)\widehat{a}_2(\varepsilon)}}, \ \varepsilon \in [0, \widehat{\varepsilon}_m].$$

(iii) The case $m > 3$. For this case the majorant equation is

$$\rho = \sum_{k=0}^{m} a_k(\varepsilon)\rho^k,$$

where the coefficients a_k are given by

$$\begin{aligned} a_0(\varepsilon) &= \varphi_m\left((\alpha_m + \varphi_m\varepsilon)^m - m\alpha_m^{m-1}\varphi_m\varepsilon - \alpha_m^m\right), \\ a_1(\varepsilon) &= m\varphi_m\left((\alpha_m + \varphi_m\varepsilon)^{m-1} - \alpha_m^{m-1}\right), \\ a_k(\varepsilon) &= \varphi_m\binom{m}{k}(\alpha_m + \varphi_m\varepsilon)^{m-k}, \ k = 2, 3, \ldots, m. \end{aligned}$$

As in the case $m = 3$, we introduce a new quadratic majorant. For this purpose we note that for the positive root $\rho_m = \rho_m(\varepsilon)$ of the equation $1 = \sum_{j=1}^{m} ja_j(\varepsilon)\rho^{j-1}$ it is fulfilled $(j+1)a_{j+1}(\varepsilon)\rho_m^j \leq 1 - a_1(\varepsilon)$. Hence,

$$a_{j+1}(\varepsilon)\rho_m^{j-1} \leq b_{j+1}(\varepsilon) := a_{j+1}^{1/j}(\varepsilon)\left(\frac{1 - a_1(\varepsilon)}{j+1}\right)^{1-1/j}, \ j = 2, 3, \ldots, m-1$$

and

$$\sum_{j=2}^{m-1} a_{j+1}(\varepsilon)\rho_m^{j-1} \leq \sum_{j=2}^{m-1} b_{j+1}(\varepsilon).$$

Thus, we may construct the quadratic Lyapunov majorant

$$\widehat{h}_m(\varepsilon, \rho) := a_0(\varepsilon) + a_1(\varepsilon)\rho + b_m(\varepsilon)\rho^2 \geq h_m(\varepsilon, \rho),$$

where

$$b_m(\varepsilon) := a_2(\varepsilon) + \sum_{j=2}^{m-1} b_{j+1}(\varepsilon).$$

Finally, we have

$$f_m(\varepsilon) \leq \widehat{f}_m(\varepsilon) := \frac{2a_0(\varepsilon)}{1 - a_1(\varepsilon) + \sqrt{(1 - a_1(\varepsilon))^2 - 4a_0(\varepsilon)b_m(\varepsilon)}}, \ \varepsilon \in [0, \widehat{\varepsilon}_m], \tag{8}$$

where $\widehat{\varepsilon}_m$ is the largest positive root of the equation $1 = a_1(\varepsilon) + 2\sqrt{a_0(\varepsilon)b_m(\varepsilon)}$. Thus, we proved the following result:

Theorem 1. *The relations* (4), (5), (7) *and* (8) *give the desired nonlocal accuracy estimates for the affine approximation of the function* $X \mapsto X^{1/m}$, $2 \leq m \in \mathbb{N}$, *at points* $X = A \in \mathbb{S}_+^{n\times n}$.

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АФИННИ АПРОКСИМАЦИИ НА РАЦИОНАЛНО–СТЕПЕННИ МАТРИЧНИ ФУНКЦИИ

Юлиана К. Бонева, Михаил М. Константинов, Владимир Т, Тодоров, Петко Хр. Петков

Намерени са нелокални строги оценки за афинните апроксимации на матричните функции $X \mapsto X^{1/m}$, където $2 \leq m \in \mathbb{N}$. Резултатите са получени с помощта на мажорантите на Ляпунов и на принципите за неподвижната точка.