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POSITIVE SOLUTIONS OF FOURTH ORDER SINGULAR BOUNDARY VALUE PROBLEMS*

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Sufficient conditions are given for existence of positive solutions of fourth order sublinear singular BVPs related to the generalized Emden-Fowler equation. A variational approach is used.

1. Introduction. In this note the boundary value problem for the fourth order equation

$$u^{(4)} = p(t)f(u), \quad 0 < t < 1 \tag{1}$$

is considered, subject to the boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0. \tag{2}$$

For the function f it is supposed to be continuous and nonnegative in $\mathbf{R}^+$, $f(0) = 0$, and that f is sublinear at 0 and at infinity. The function p is positive and continuous on $(0,1)$, and p may grow to infinity at $t = 0$ and $t = 1$. The motivation for studying such problems is due to their applications. For example, the deformation of an elastic beam with clamped ends in equilibrium state can be described by a fourth order BVP of that type.

It is worth indicating here that the nontrivial solution u of (1), (2) must be positive, i.e. $u > 0$ on $(0,1)$. Indeed, for any nonzero solution $u \in C^1[0,1] \cap C^4(0,1)$, by the equation it follows that u'' is convex. Also, by the boundary conditions we have that both u' and u'' have zeros in $(0,1)$. Indeed, suppose that $u'' \geq 0$ on $(0,1)$. Then, u' increases on $[0,1]$ which is a contradiction. Now, the question is how many (simple) zeros u'' has: one or two. Suppose u'' has only one (simple) zero. Then, u' does not possess zeros on $(0,1)$, and we come to a contradiction again. Finally, there are $0 < t_1 < t_0 < t_2 < 1$ such that $u'(t_0) = u''(t_1) = u''(t_2) = 0$ which combined with the convexity of u'' means that $u > 0$ on $(0,1)$.

The model equation of (1) is

$$u^{(4)} = p(t)u^{\lambda}, \quad 0 < t < 1 \tag{3}$$

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where $\lambda \in (0,1)$ is given. The problem (3), (2) has been studied recently by Ma & Tisdell [1] and Cui & Zou [2] *via* the method of lower and upper solutions and fixed point index theorem. In [1] the authors have shown that necessary and sufficient condition for existence of positive solutions $u \in C^2[0,1] \cap C^4(0,1)$ is

$$0 < \int_0^1 t^{1+2\lambda} (1-t)^{1+2\lambda} p(t)\, dt < \infty, \tag{4}$$

and if

$$0 < \int_0^1 t^2 (1-t)^2 p(t)\, dt < \infty \tag{5}$$

is satisfied, then (3), (2) has positive solutions $u \in C^1[0,1] \cap C^4(0,1)$.

In this note a sufficient condition is obtained for existence of positive solutions $u \in H_0^2(0,1)$ of the problem (1), (2). As a corollary we have that if

$$0 < \int_0^1 t^{\frac{3}{2}(1+\lambda)} (1-t)^{\frac{3}{2}(1+\lambda)} p(t)\, dt < \infty, \tag{6}$$

then (3), (2) has positive solutions $u \in H_0^2(0,1)$. Since $0 < \lambda < 1$, the condition (4) implies (6) which is natural because

$$C^2[0,1] \subset H^2(0,1) \subset C^1[0,1],$$

by the embedding theorem. On the other hand, if $0 < \lambda \le \frac{1}{3}$, then (6) implies (5), i.e. our result is between those of Ma & Tisdell [1] in that case. However, if $\frac{1}{3} < \lambda < 1$, then (5) implies (6) which means that the condition (6) is better than (5).

2. Existence results. In this section we consider the boundary value problem

$$u^{(4)} = p(t)f(u), \quad 0 < t < 1, \tag{7}$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where p and f satisfy the following assumptions:

(1) $p \in C(0,1)$, $p > 0$ on $(0,1)$,

(2) $f \in C(\mathbf{R}^+, \mathbf{R}^+)$, $f(0) = 0$,

(3) for some λ, $0 < \lambda < 1$,

(*i*) $0 < \int_0^1 (s(1-s))^{\frac{3(1+\lambda)}{2}} p(s)\, ds < \infty$,

(*ii*) $0 < \liminf_{u \to 0^+} \frac{f(u)}{u^\lambda} \le \limsup_{u \to 0^+} \frac{f(u)}{u^\lambda} < \infty$,

(4) $\lim_{u \to +\infty} \frac{f(u)}{u} = 0$.

The main result is:

Theorem 1. *Suppose that the conditions* $(1)-(4)$ *are satisfied. Then,* (7) *has positive solutions* $u \in H_0^2(0,1)$.

We begin the proof with the following

Lemma 2. *Under the conditions* (1), (3*i*), *the space* $H_0^2(0,1)$ *is embedded continuously into the Banach space*

$$L_p^{1+\lambda}(0,1) := \left\{ u : \int_0^1 p(s)\,|u(s)|^{1+\lambda}\,ds < \infty \right\}$$

with the norm $\|u\|_p = \left(\int_0^1 p(s)\,|u(s)|^{1+\lambda}\,ds \right)^{\frac{1}{1+\lambda}}$.

Proof. Let us mention first that

$$\|u\| = \left(\int_0^1 u''^2 dt \right)^{\frac{1}{2}}$$

is an equivalent norm in $H_0^2(0,1)$, since

$$\int_0^1 u'^2 dt \le \frac{1}{2}\int_0^1 u^2 dt + \frac{1}{2}\int_0^1 u''^2 dt, \qquad \int_0^1 u^2 dt \le \frac{1}{\pi^4}\int_0^1 u''^2 dt, \qquad u \in H_0^2(0,1).$$

For $u \in H_0^2(0,1)$ we have

$$u(t) = \int_0^t \int_0^s u''(\tau)\,d\tau ds = \int_0^t (t-\tau)\,u''(\tau)\,d\tau, \quad t \in [0,1].$$

Thus,

$$|u(t)| \le \|u\| \left(\int_0^t (t-\tau)^2\,d\tau \right)^{\frac{1}{2}} = \frac{t^{\frac{3}{2}}}{\sqrt{3}}\,\|u\|, \quad t \in [0,1].$$

In the same way for $u \in H_0^2(0,1)$ one has

$$|u(t)| \le \frac{(1-t)^{\frac{3}{2}}}{\sqrt{3}}\,\|u\|, \quad t \in [0,1].$$

Consequently,

$$\begin{aligned} &(1-t)^{\frac{3}{2}(1+\lambda)} \int_0^t p(s)\,|u(s)|^{1+\lambda} \\ \le\ & 3^{-\frac{1+\lambda}{2}} \left(\int_0^t (s(1-s))^{\frac{3}{2}(1+\lambda)}\,p(s)\,ds \right) \|u\|^{1+\lambda}, \quad t \in [0,1), \\ &t^{\frac{3}{2}(1+\lambda)} \int_t^1 p(s)\,|u(s)|^{1+\lambda} \\ \le\ & 3^{-\frac{1+\lambda}{2}} \left(\int_t^1 (s(1-s))^{\frac{3}{2}(1+\lambda)}\,p(s)\,ds \right) \|u\|^{1+\lambda}, \quad t \in (0,1]. \end{aligned}$$

Choosing $t = \frac{1}{2}$, the last two inequalities yield

$$\|u\|_p^{1+\lambda} \le \left(\frac{8}{3}\right)^{\frac{1+\lambda}{2}} \left(\int_0^1 (s(1-s))^{\frac{3}{2}(1+\lambda)}\,p(s)\,ds \right) \|u\|^{1+\lambda}$$

which completes the proof. □

Lemma 3. *Under the hypotheses of Lemma* 2, *the embedding of* $H_0^2(0,1)$ *into* $L_p^{1+\lambda}(0,1)$ *is compact.*

Proof. Let (u_k) be a sequence which is weakly convergent to 0 in $H_0^2(0,1)$. Then, there exists $c > 0$ such that

$$\|u_k\| \le c, \quad \forall k. \tag{8}$$

Since (u_k) is uniformly convergent to 0 in $[0,1]$, for $\varepsilon > 0$ there is a number N such that $|u_n(t)| < \varepsilon$ for all $n > N$ and all $t \in [0,1]$.

By the absolute continuity of the Lebesgue integral, there are $0 < \delta_1 < \frac{1}{2} < \delta_2 < 1$ such that

$$\int_0^{\delta_1} (s(1-s))^{\frac{3(1+\lambda)}{2}} p(s)\, ds < \varepsilon, \qquad \int_{\delta_2}^{1} (s(1-s))^{\frac{3(1+\lambda)}{2}} p(s)\, ds < \varepsilon.$$

Then, by (8) we have

$$\begin{aligned}
\left(\frac{1}{2}\right)^{\frac{3}{2}(1+\lambda)} \int_0^{\delta_1} p(s)\, |u_n(s)|^{1+\lambda}\, ds &\le (1-\delta_1)^{\frac{3}{2}(1+\lambda)} \int_0^{\delta_1} p(s)\, |u_n(s)|^{1+\lambda}\, ds \\
&\le 3^{-\frac{1+\lambda}{2}} c^{1+\lambda} \int_0^{\delta_1} (s(1-s))^{\frac{3(1+\lambda)}{2}} p(s)\, ds \\
&< 3^{-\frac{1+\lambda}{2}} c^{1+\lambda} \varepsilon, \\
\left(\frac{1}{2}\right)^{\frac{3}{2}(1+\lambda)} \int_{\delta_2}^{1} p(s)\, |u_n(s)|^{1+\lambda}\, ds &\le \delta_2^{\frac{3}{2}(1+\lambda)} \int_{\delta_2}^{1} p(s)\, |u_n(s)|^{1+\lambda}\, ds \\
&\le 3^{-\frac{1+\lambda}{2}} c^{1+\lambda} \int_{\delta_2}^{1} (s(1-s))^{\frac{3(1+\lambda)}{2}} p(s)\, ds \\
&< 3^{-\frac{1+\lambda}{2}} c^{1+\lambda} \varepsilon.
\end{aligned}$$

Consequently,

$$\int_0^{\delta_1} p(s)\, |u_n(s)|^{1+\lambda}\, ds \to 0, \qquad \int_{\delta_2}^{1} p(s)\, |u_n(s)|^{1+\lambda}\, ds \to 0.$$

On the other hand

$$\int_{\delta_1}^{\delta_2} p(s)\, |u_n(s)|^{1+\lambda}\, ds \to 0,$$

and the proof is complete. □

Now, we are ready to establish Theorem 1. We put the problem (7) in a variational setting by introducing the functional

$$J(u) = \int_0^1 \left(\frac{1}{2} u''^2 - p(t)\, \overline{F}(u)\right) dt$$

with $\overline{F}(u) = \int_0^u \overline{f}(s)\, ds$ and $\overline{f}(u)$ defined by $\overline{f}(u) = 0$ for $u < 0$, $\overline{f}(u) = f(u)$ for $u \ge 0$. As in [3], Theorem 1, it can be shown that J is bounded from below, coercive and weakly lower semicontinuous in $H_0^2(0,1)$. Then, by the general minimization theorem (cf. [4], Theorem), J has a minimizer which is a solution of (7). Moreover, since f is sublinear near 0, the minimizer of J is nontrivial, i.e. the problem (7) possesses positive solution.

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ПОЛОЖИТЕЛНИ РЕШЕНИЯ НА СИНГУЛЯРНИ ГРАНИЧНИ ЗАДАЧИ ОТ ЧЕТВЪРТИ РЕД

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Получено е достатъчно условие за съществуване на положително решение на сингулярна сублинейна гранична задача от четвърти ред, свързана с обобщеното уравнение на Емден-Фоулър. Използван е вариационен подход.