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SOME RESULTS ABOUT CONVERGENCE AND SUMMABILITY OF SERIES IN HERMITE ASSOCIATED FUNCTIONS*

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In this paper some results about convergence and Cesaro summability of series in Hermite associated functions are considered.

Hermite polynomials are defined by the equalities [1, (2.11), p. 12]

$$H_n(z) = (-1)^n \exp(z^2)\{\exp(-z^2)\}^{(n)} \qquad (n = 0,\ 1,\ 2,).$$

The functions $\{G_n(z)\}_{n=0}^{\infty}$, defined in the open set $H = C\backslash R$ by means of the equalities [1, (4.13), p.27]

$$G_n(z) = -\int_{-\infty}^{+\infty} \frac{\exp(-t^2)H_n(t)}{t-z}dt, \quad n = 0, 1, 2, \dots,$$

are called Hermite associated functions. These functions are holomorphic in the open set H.

Now, we define the following two sequences of holomorphic functions:

(1) $$G_n^+(z) = G_n(z), \quad \operatorname{Im} z > 0, \quad n = 0, 1, 2, \dots,$$

and

(2) $$G_n^-(z) = G_n(z), \quad \operatorname{Im} z < 0, \quad n = 0, 1, 2, \dots$$

For the Hermite associated functions (1) and (2) the following proposition is true [1, (III.3.4)]:

(a) The representation

(3) $$G_n^+(z) = \pi\sqrt{2}(-i)^{n+1}(2n/e)^{n/2}\exp(-z^2/2)\exp(iz\sqrt{2n+1})[1+k_n^+(z)], \quad n = 1, 2, \dots$$

holds in the half-plane $H^+ : \operatorname{Im} z > 0$, where the complex functions $\{k_n^+(z)\}_{n=1}^{+\infty}$ are holomorphic in H^+ and $k_n^+(z) = o(1)(n \to \infty)$ uniformly on every compact subset of H^+.

(b) The representation

(4) $$G_n^-(z) = i^{n+1}\pi\sqrt{2}(2n/e)^{n/2}\exp(-z^2/2)\exp(-iz\sqrt{2n+1})[1+k_n^-(z)], \quad n = 1, 2, \dots$$

holds in the half-plane $H^- : \operatorname{Im} z < 0$, where the complex functions $\{k_n^-(z)\}_{n=1}^{+\infty}$ are ho-lomorphic in H^- and $k_n^-(z) = o(1)(n \to \infty)$ uniformly on every compact subset of H^-.

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It holds that $k^-(z) = \overline{k_n^+(\overline{z})}$, $n = 1, 2, 3, \ldots$.

A series of the kind

$$\sum_{n=0}^{+\infty} a_n G_n^{\pm}(z) \tag{5}$$

we call Hermite series.

Let $0 < \tau < +\infty$ and $S(\tau) = \{z \in C : |\operatorname{Im} z| > \tau\}$. We assume that $S(0) = H$ and $S(\infty) = \emptyset$.

P. Rusev proved the following assertion [1, (IV.3.3), p. 99]

Theorem 1. (*a*) *If the Hermite series* (5) *converges at a point* $z_0 \in H$*, then it is absolutely uniformly convergent on every closed set* $\overline{S(\tau)}$ *with* $\tau > |\operatorname{Im} z_0|$.

(*b*) *If*

$$\tau_0 = \max\{0, \lim_{n\to\infty} \sup(2n+1)^{-1} \log|(2n/e)^{n/2} a_n\},$$

then for $\tau \in (\tau_0, \infty)$ *the Hermite series* (5) *is absolutely uniformly convergent on the closed set* $\overline{S(\tau)}$*and diverges in* $C\backslash\overline{S(\tau_0)}$.

The main result in this paper is the following:

Theorem 2. *Let* $p \geq -1$, $z_0 \in H$ *and*

$$a_n G_n^{\pm}(z_0) = O(n^p)(n \to +\infty). \tag{6}$$

Then, the Hermite series (5) *is absolutely convergent in the set* $S(\tau_0)$*, where* $\tau_0 = |\operatorname{Im} z_0|$.

Proof. Suppose that $z_0 = x_0 + \tau_0 i \in H^+$ and $z_1 = x_1 + y_1 i \in S(\tau_0) \cap H^+$. Then,

$$y_1 > \tau_0. \tag{7}$$

We shall prove that the series (5) is absolutely convergent for $z = z_1$.

Using (3), it is not difficult to prove that

$$|G_n^+(z)| = K_n O(\exp(-\sqrt{2n+1}y) \qquad (n \to +\infty), \tag{8}$$

where $K_n = (2n/e)^{n/2}$, $z \in H^+$ and $y = \operatorname{Im} z$.

We assume that $a_0 = 0$. Using the representation (3) it is not difficult to prove that there is number $m \in \mathbb{N}$ such that $G_n^+(z_0) \neq 0$ for $n \geq m$.

Suppose that $n \geq m$. Then,

$$b_n = a_n G_n^+(z_1) = a_n G_n^+(z_0).\frac{G_n^+(z_1)}{G_n^+(z_0)}.$$

Having in mind the asymptotic formula (8), we get that

$$\frac{G_n^+(z_1)}{G_n^+(z_0)} = O(\exp(-\lambda\sqrt{2n+1})) \qquad (n \to +\infty),$$

where $\lambda = \tau_0 - y_1$. From inequality (7) it follows that $\lambda > 0$.

Using (6), we get that

$$\sum_{n=m}^{+\infty} |b_n| = O\left(\sum_{n=m}^{+\infty} n^p \exp(-\lambda\sqrt{2n+1})\right).$$

Since the series $\sum_{n=1}^{+\infty} n^p \exp(-\lambda\sqrt{2n+1})$ converges, we conclude that the series $\sum_{n=1}^{+\infty} |b_n|$ is convergent. Therefore, the series (5) is absolutely convergent for $z = z_1$.

Suppose that $z_2 = x_2 - y_2 i \in S(\tau_0) \cap H^-$. Then,

$$y_2 > \tau_0.$$

We shall prove that the series (5) is absolutely convergent for $z = z_2$.

Using (4), it is easy to prove that

(9) $$|G_n^-(z)| = K_n O(\exp(\sqrt{2n+1} y) \qquad (n \to +\infty)),$$

where $z \in H^-$ and $y = \operatorname{Im} z$.

Suppose that $n \geq m$. Then,

$$c_n = a_n G_n^-(z_2) = a_n G_n^+(z_0). \frac{G_n^-(z_2)}{G_n^+(z_0)}.$$

Using (8) and (9), we get that

$$\frac{G_n^-(z_2)}{G_n^+(z_0)} = O(\exp(-\mu\sqrt{2n+1})) \qquad (n \to +\infty),$$

where $\mu = y_2 - \tau_0 > 0$. Then, we have that

$$\sum_{n=m}^{+\infty} |c_n| = O\left(\sum_{n=m}^{+\infty} n^p \exp(-\mu\sqrt{2n+1})\right).$$

Hence, the series (5) is absolutely convergent for $z = z_2$.

The proof of Theorem 2 in the case when $z_0 \in H^-$ is similar to the proof in the case when $z_0 \in H^+$. Thus, Theorem 2 is proved. □

As a corollary of Theorem 1 (a) and Theorem 2 we can state the following proposition:

Theorem 3. *Let $p \geq -1$, $z_0 \in H$ and let* (6) *hold. Then, the Hermite series* (5) *is absolutely uniformly convergent on every closed set $\overline{S(\tau)}$ with $\tau > |\operatorname{Im} z_0|$.*

Let the series (5) be Cesaro summability with parameter $\delta > -1$, i.e. (C, δ) – summable for $z = z_0 \in H$. Then [2, p. 132]

$$a_n G_n^{\pm}(z_0) = O(n^{\delta}) \qquad (n \to +\infty).$$

Applying Theorem 2 we get that the series (5) is convergent for $z \in S(|\operatorname{Im} z_0|)$.

Then, as another corollary of Theorem 2 we get the following result:

Theorem 4. *Let $\delta > -1$ and let the series* (5) *be (C, δ)-summable for $z = z_0 \in H$. Then, the Hermite series* (3) *is absolutely convergent in the set $S(|\operatorname{Im} z_0|)$.*

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НЯКОИ РЕЗУЛТАТИ ЗА СХОДИМОСТТА И СУМИРУЕМОСТТА НА РЕДОВЕ ПО АСОЦИИРАНИТЕ ФУНКЦИИ НА ЕРМИТ

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В този статия са разгледани някои твърдения, свързани със сходимостта и сумируемостта на редове по асоциираните функции на Ермит.