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ON KENMOTSU TYPE PARAMETERIZATION OF DELAUNAY SURFACES*

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The equilibrium conditions for a patch on an axisymmetric membranes could be reduced to the defining equation of the constant mean curvature surfaces of revolution. Solving it, an explicit parameterization of these surfaces along various formulas regarding their geometry have been derived and the corresponding surfaces were depicted as functions of two appropriately chosen real parameters.

1. Introduction. Some years ago Kenmotsu [11] found and solved a complex nonlinear differential equation which describes the surfaces of revolution in $\mathbb{R}^3$ with a given mean curvature. The subclass of the surfaces of constant mean curvature have been introduced long time before by the French geometer Delaunay [2], who gave also their complete classification. The list includes planes, cylinders, catenoids, spheres, nodoids and unduloids.

It should be noted also that the methods used by Delaunay and Kenmotsu are of completely different nature. The former relays on purely geometrical ideas and constructions while the latter represents the generating curves of the surfaces in terms of the generalized Fresnel's integrals built on the specified mean curvature. Another important moment is that in both approaches the final formulas are given as integrals which even are not well defined in the case of Delaunay's parameterization while the parameters entering in Kenmotsu representation of constant mean curvature surfaces of revolution have not quite clear geometrical interpretation.

Recently, it has been realized that there exists another purely mechanical approach to Delaunay's surfaces based on consideration of the equilibrium conditions for axisymmetric membranes [7]. A specific choice about quantities that play the most crucial role in this sort of problems like weight density w of the film, circumferential σ_c and meridional σ_m stresses and the differential pressure p across the membrane surface singles out exactly the case of the Delaunay surfaces. Here this case is solved up to the very end and the relevant results are given in an analytical form.

2. Surfaces of revolution in $\mathbb{R}^3$. As usual we think about the axisymmetric surface $\mathcal{S}$ in the ordinary Euclidean space by specifying its meridional section, i.e., a

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curve $u \longrightarrow (r(u), z(u))$ in the XOZ plane, assuming that u is the so called natural parameter provided by the corresponding arclength. The surface $\mathcal{S}$ in $\mathbb{R}^3$ with a fixed orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ can be presented by making use of the parameter u and the angle v specifying the rotation of the XOY plane *via* the vector-valued function

$$\mathbf{x}(u,v) = r(u)\mathbf{e}_1(v) + z(u)\mathbf{e}_3(v), \qquad u \in \mathbb{R}, \quad 0 \le v < 2\pi. \tag{1}$$

Here, the vector $\mathbf{e}_1(v)$ is the new position of $\mathbf{i}$ after the rotation at angle v, i.e.

$$\mathbf{e}_1(v) = \cos v\mathbf{i} + \sin v\mathbf{j}. \tag{2}$$

The tangent vector at each point of the generating curve is given by the first derivative with respect to u:

$$\mathbf{t}(u,v) = \mathbf{x}_u(u,v) = r'(u)\mathbf{e}_1(v) + z'(u)\mathbf{k}. \tag{3}$$

In equation (3), and elsewhere in this paper, the prime denotes the derivative with respect to the meridional arc length u. Let us introduce also the function $\theta(u)$ which measures the angle between the normal vector $\mathbf{n}$ to $\mathcal{S}$ and $\mathbf{k}$. Then, the coordinates $r(u)$ and $z(u)$ depend on $\theta(u)$ through the equations

$$r'(u) = \cos\theta(u) \tag{4}$$

$$z'(u) = -\sin\theta(u). \tag{5}$$

Using the standard formulas from the textbooks on classical differential geometry (see, e.g. [9, 16, 19]) one can easily find the so called principal curvatures along meridional, respectively, parallel directions, i.e.,

$$k_\mu = -\theta'(u) \qquad \text{and} \qquad k_\pi = -\frac{\sin\theta(u)}{r(u)} \tag{6}$$

and after that the mean (meaning average) curvature H

$$H = \frac{1}{2}(k_\mu + k_\pi) = -\frac{1}{2}\left(\theta'(u) + \frac{\sin\theta(u)}{r(u)}\right).$$

3. Equilibrium equations and shapes. By balancing the internal and external forces acting on the axisymmetric membrane (which are vectors) and projecting them onto $\mathbf{n}$ and $\mathbf{t}$, respectively, we obtain the following equilibrium equations

$$(\sigma_m(u)r(u))\frac{\mathrm{d}\theta(u)}{\mathrm{d}u} = -w(u)r(u)\cos\theta(u) - \sigma_c(u)\sin\theta(u) + p(u)r(u) \tag{7}$$

$$\frac{\mathrm{d}(\sigma_m(u)r(u))}{\mathrm{d}u} = -w(u)r(u)\sin\theta(u) + \sigma_c(u)\cos\theta(u). \tag{8}$$

Let us consider the case of a membrane for which the film weight contribution can be neglected, i.e. $w(u) \equiv 0$ and, therefore, we have instead the equations (7) and (8), the system

$$-(\sigma_m(u)r(u))\frac{\mathrm{d}\theta(u)}{\mathrm{d}u} = \sigma_c(u)\sin\theta(u) - p(u)r(u), \tag{9}$$

$$\frac{\mathrm{d}(\sigma_m(u)r(u))}{\mathrm{d}u} = \sigma_c(u)\cos\theta(u). \tag{10}$$

The geometrical relation (4) and the second equation in this system imply that the meridional and circumferential stresses are constant and of the same magnitude, i.e. $\sigma_m(u) = \sigma_c(u) = \sigma = \text{constant}$, while the first equation (9) can be recognized as the mean curvature of $\mathcal{S}$, namely,

$$H = -\frac{p(u)}{2\sigma} \cdot \tag{11}$$

If we continue with examination of the case where the hydrostatic pressure is also a constant, i.e. $p(u) = p_o =$ constant, then we end up with a surface of constant mean curvature

$$H = -\frac{p_o}{2\sigma} = -\frac{\mathring{p}}{2} = \text{constant}, \qquad \mathring{p} = \frac{p_o}{\sigma} \cdot \tag{12}$$

Following a genuine geometrical argument Delaunay [2] generates the corresponding profile curves as traces of the foci of the non-degenerate conics when they roll along a straight line in the plane (*roulettes* in French). On the other hand, these surfaces of revolution have a minimal lateral area at a fixed volume as it is proved by Sturm in an Appendix to the same paper. That in turn reveals why these surfaces make their appearance as soap bubbles and liquid drops [4, 8, 15] in Physics or myelin shapes [3] and cells under compression [6, 20] in Biology.

4. Delaunay surfaces. In this section we present the analytical description of the surfaces from Delaunay's list. We start with the system formed by the equations (9) and (10)complemented by the assumption $p(u) = p_0 =$ const which taken together ensures the geometrical relation

$$\sin\theta(u) = \frac{\mathring{p}\, r}{2} + \frac{C}{r}, \tag{13}$$

where C is some integration constant. Without any loss of generality we can assume also that $\mathring{p}$ is a positive number relying either to physical experiments with membranes and balloons, or taking into account the mathematical fact that r is always positive and that we can measure $\theta(u)$ only in two ways – clockwise or counterclockwise, but we do not take advantage of this assumption in our considerations. Actually, we have to consider the following cases:

1. Both numerical parameters $\mathring{p}$ and C in (13) are zero. This means that $\theta(u)$ is identically zero as well and, therefore, the surfaces under consideration are parallel Euclidean planes.

2. The parameter $\mathring{p}$ is zero and the integration constant C is non-zero. As $\mathring{p}$ is directly related to the mean curvature of the surface, this means that we are dealing with a rotational surfaces of zero mean curvature. According to the well-knowna theorem in the classical differential geometry (see [9, 16, 19]), this class of surfaces is exhausted by the catenoids. See also [12].

3. Both constants $\mathring{p}$ and C are non-zero and satisfy the condition $\mathring{p}\, C \leq 1/2$. Elsewhere (cf. [7]) it has been proven that in this case

$$r^2(u) = m \sin\mu u + n, \tag{14}$$

where

$$m = (c^2 - a^2)/2, \qquad n = (c^2 + a^2)/2, \qquad \mu = \mathring{p} = \frac{2}{a+c}, \tag{15}$$

and in order to make a contact with the paper mention above, we have to remind also the relations

$$(16) \qquad a = \frac{1-\sqrt{1-2\mathring{p}C}}{\mathring{p}}, \qquad c = \frac{1+\sqrt{1-2\mathring{p}C}}{\mathring{p}}, \qquad C = \frac{ac}{a+c}\cdot$$

In the new variables the constraint $\mathring{p}C \leq 1/2$ reads

$$(17) \qquad |a-c| \geq 0$$

which is obviously fulfilled for all real numbers a and c and, therefore, (17) does not impose any restrictions on the parameters. The only restriction which has to be taken into account is that at the left hand side of (13) we have a bounded function and this implies that the function on the right should be of the same type. However, this is equivalent to the requirement that the coefficients $\mathring{p}$ and C are finite. Looking at (15) and (16), it is easy to realize that this is always the case provided $a+c \neq 0$ and, hence, this is the only restriction which a and c must obey.

	$a<0$	$a>0$	$a=c$	$a=0$
$c<0$	Unduloids	Nodoids	Cylinders	Spheres
$c>0$	Nodoids	Unduloids	Cylinders	Spheres

Table 1. Various Delaunay surfaces corresponding to different ranges of values of the parameters a and c

After clarifying this point we can continue with determining the profile curve. For that purpose we can combine the geometrical relations (5) and (13) and make use of the notation introduced above in order to write down the equation

$$(18) \qquad z'(u) = -\sin\theta(u) = -\frac{1}{a+c}\left[r(u) + \frac{ac}{r(u)}\right]$$

which gives us immediately

$$(19) \qquad z(u) = -\frac{1}{a+c}\left[\int r(\tilde{u})\mathrm{d}\tilde{u} + ac\int\frac{\mathrm{d}\tilde{u}}{r(\tilde{u})}\right].$$

The two integrals on the right can be converted into standard elliptic integrals of the second, respectively, first kind by performing the changes

$$(20) \qquad \sin\mu\tilde{u} = 1 - 2\mathrm{sn}^2(t,k), \qquad \mathrm{d}\tilde{u} = -(a+c)\mathrm{dn}(t,k)\mathrm{d}t,$$

where $\mathrm{sn}(t,k)$ and $\mathrm{dn}(t,k)$ are two of the Jacobian elliptic functions of the argument t and the *elliptic module* k (details about elliptic functions, their integrals, and properties can be found in [10]). Choosing $k^2 = 2m/(m+n) = (c^2-a^2)/c^2$, we end up with a linear combination of the first and second kind elliptic integrals $F(\varphi,k)$ and $E(\varphi,k)$ in their canonical forms, namely

$$(21) \qquad z(t) = c\int \mathrm{dn}^2(t,k)\mathrm{d}t + a\int \mathrm{d}t = c\,E\left(\mathrm{am}(t,k),k\right) + aF\left(\mathrm{am}(t,k),k\right).$$

A little bit more work shows that the Jacobian amplitude function $\mathrm{am}(t,k)$ which appears above can be replaced by $\pi/4 - \mu u/2$, so that the profile curve has a parameterization

$$(22) \qquad r(u) = \sqrt{m\sin\mu u + n}, \quad z(u) = c\,E\left(\frac{\pi}{4} - \frac{\mu u}{2}, k\right) + aF\left(\frac{\pi}{4} - \frac{\mu u}{2}, k\right).$$

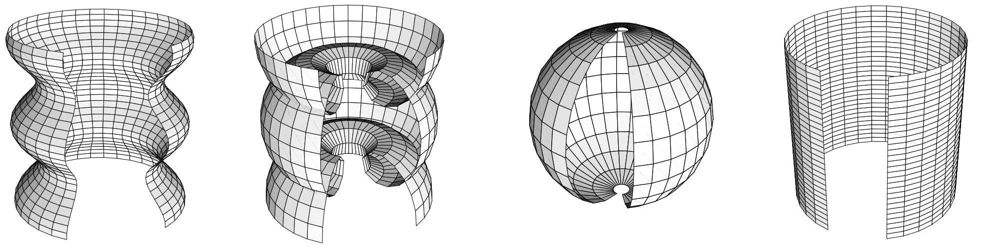

Fig. 1. The open parts of the unduloid, nodoid, sphere and cylinder shown here, are drawn using the profile curve (22) and various combinations of the parameters a and c according to the specifications in Table 1

5. Some useful formulas. Having the explicit form of the parameterization which is generated by the meridional section of the surface one can easily find the corresponding geometrical characteristics like the mean, Gauss and principal curvatures. For such a purpose one needs only the first and second fundamental forms of the surface under consideration. In our case these forms are

$$I = \mathrm{d}u^2 + \frac{1}{2}(a^2 + c^2 + \left(c^2 - a^2\right)\sin\frac{2u}{a+c})\,\mathrm{d}v^2 \tag{23}$$

$$II = -\frac{(c-a)\Big(c - a + (a+c)\sin\frac{2u}{a+c}\Big)}{(a+c)\Big(a^2 + c^2 + (c^2 - a^2)\sin\frac{2u}{a+c}\Big)}\,\mathrm{d}u^2 - \frac{1}{2}\left(a + c + (c-a)\sin\frac{2u}{a+c}\right)\mathrm{d}v^2 \tag{24}$$

and we have, respectively, also

$$k_\mu = -\frac{(c-a)\ (c - a + (a+c)\ \sin\frac{2u}{a+c}\ )}{(a+c)\ (a^2 + c^2 + (c^2 - a^2)\ \sin\frac{2u}{a+c}\ )} \tag{25}$$

$$k_\pi = -\frac{a + c + (c-a)\sin\frac{2u}{a+c}}{a^2 + c^2 + (c^2 - a^2)\sin\frac{2u}{a+c}}\,. \tag{26}$$

These formulas allows an easy check that taking $a = 0$, $c = 0$, or $c = a$, one indeed gets either spheres or a cylinder in a full agreement with the classification presented above in Table 1. The formulas above are indispensable also in the problems like finding the length, surface area or the volume inside the Delaunay surfaces [5]. We hope to report about these problems elsewhere.

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ВЪРХУ ПАРАМЕТРИЗАЦИИТЕ ОТ ТИПА НА KENMOTSU НА ПОВЪРХНИНИТЕ НА DELAUNAY

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Условията за равновесие на една аксиално симетрична мембрана могат да бъдат редуцирани до уравнението за постоянна средна кривина на повърхнината която я определя. Намерени са явните решения на това уравнение като функции на два подходящо избрани реални параметъра, които пораждат параметризации от типа на Kenmotsu на ротационните повърхнини с постоянна средна кривина. Споменатите параметри характеризират съответните повърхнини и конкретните им значения фиксират графичната им реализация.