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DOUBLE-COMPLEX DIFFERENTIAL FORMS*

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Double-complex function theory [1, 2, 3] is an alternative isomorphic version of the former bi-complex function theory initiated by C. Segre [4]. The double-complex numbers, elements of the double-complex algebra, denoted $\mathbf{C}(1,j)$, are represented as follows: $\alpha = z + jw$, where $j^2 = i$, and z, w are complex numbers. The algebra $\mathbf{C}(1,j)$ is not a division algebra. A complex-analytic structire can be defined by an analogue of the Cauchy-Riemann equations. In this note we develop some basic notions of differential forms on $\mathbf{C}(1,j)$ and we study different quadratic geometries (Q-geometries [6]) over the double-complex algebra $\mathbf{C}(1,j)$, and over the bi-complex algebra $\mathbf{BC}$ [5]. In fact, these two algebras are isomorphic.

1. We consider differential 1-forms $\omega = \varphi(\alpha)d\alpha + \psi(a)da^*$ on the double-complex algebra $\mathbf{C}(1,j)$. These 1-forms generalize the formula for the differential of a double-complex function $f(\alpha) = f_0(z,w) + jf_1(z,w)$, namely

$$df = \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \alpha^*} d\alpha^*,$$

where $\alpha^* = z - jw$ is the conjugate of $\alpha = z + jw$ and

$$\frac{\partial f}{\partial \alpha} = \frac{1}{2}\left(\frac{\partial f_0}{\partial z} + \frac{\partial f_1}{\partial w}\right) - j\frac{i}{2}\left(\frac{\partial f_0}{\partial w} + i\frac{\partial f_1}{\partial z}\right),$$

$$\frac{\partial f}{\partial \alpha^*} = \frac{1}{2}\left(\frac{\partial f_0}{\partial z} - \frac{\partial f_1}{\partial w}\right) + j\frac{i}{2}\left(\frac{\partial f_0}{\partial w} - i\frac{\partial f_1}{\partial z}\right),$$

$$d\alpha = dz + j \quad d\alpha^* = dz - jdw.$$

The operator of exterior differentiation is defined as usually:

$$d\omega = d\varphi(\alpha) \wedge d\alpha + d\psi(\alpha) \wedge d\alpha^*.$$

By definition, $d\alpha \wedge d\alpha = d\alpha^* \wedge d\alpha^* = 0$, and $d\alpha \wedge d\alpha^* = -d\alpha^* \wedge d\alpha$.

It is not difficult to see that $d^2\omega = 0$ for double-complex differential forms.

Calculating, we obtain:

$$d\omega = \left(\frac{\partial \psi}{\partial \alpha} - \frac{\partial \varphi}{\partial \alpha^*}\right) d\alpha \wedge d\alpha^* \quad \text{and} \quad d\alpha \wedge d\alpha^* = -2jdz \wedge dw.$$

Differential 2-forms $\Omega = F(\alpha)d\alpha \wedge d\alpha^*$ are defined as usually and, respectively, for the operator of exterior differentiation d we have always $d\,\Omega = 0$.

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Clearly, each 2-form is closed, and the 1-form $\omega = \varphi(\alpha)d\alpha + \psi(a)da^*$ is closed iff:

$$\frac{\partial\psi}{\partial\alpha} - \frac{\partial\varphi}{\partial\alpha^*} = 0.$$

Having in mind that:

$$\psi(\alpha) = \psi_0(z,w) + j\psi_1(z,w) \quad \text{and} \quad \varphi(\alpha) = \varphi_0(z,w) + j\varphi_1(z,w),$$

$$\frac{\partial\psi}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial\psi_0}{\partial z} + \frac{\partial\psi_1}{\partial w}\right) - j\frac{i}{2}\left(\frac{\partial\psi_0}{\partial w} + i\frac{\partial\psi_1}{\partial z}\right),$$

$$\frac{\partial\varphi}{\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial z} - \frac{\partial\varphi_1}{\partial w}\right) + j\frac{i}{2}\left(\frac{\partial\varphi_0}{\partial w} - i\frac{\partial\varphi_1}{\partial z}\right),$$

we express the last equation in terms of partial complex derivatives $\frac{\partial\psi_0}{\partial z}$, $\frac{\partial\varphi_0}{\partial z}$, $\frac{\partial\psi_1}{\partial w}$, $\frac{\partial\varphi_1}{\partial w}$. As a result we obtain that the 1-form ω is closed iff:

$$\frac{\partial(\psi_0 - \varphi_0)}{\partial z} + \frac{\partial(\psi_1 + \varphi_1)}{\partial w} = 0,$$

$$-\frac{\partial(\psi_1 - \varphi_1)}{\partial z} + i\frac{\partial(\psi_0 + \varphi_0)}{\partial w} = 0.$$

The last remark does not concern the holomorphic differential 1-forms on the considered algebra, i.e. the forms $\omega = \varphi(\alpha)d\alpha$ with holomorphic coefficient $\varphi(\alpha)$. This means that $\partial\varphi/\partial\alpha^* = 0$, or, equivalently, the Cauchy-Riemann double-complex system is valid for $\varphi(\alpha) = \varphi_0(z,w) + j\varphi_1(z,w)$:

$$\frac{\partial\varphi_0}{\partial z} = \frac{\partial\varphi_1}{\partial w}, \quad \frac{\partial\varphi_0}{\partial w} = i\frac{\partial\varphi_1}{\partial z}. \tag{1}$$

For detailed exposition see [1] or [2].

Each holomorphic 1-form is closed. Indeed, we have:

$$d\omega = d\varphi(\alpha) \wedge d\alpha = \left(\frac{\partial\varphi}{\partial\alpha}d\alpha + \frac{\partial\varphi}{\partial\alpha^*}d\alpha^*\right) \wedge d\alpha = 0.$$

We say that the double-complex differential 1-form ω is *exact* in the domain $G \subset \mathbf{C}\times\mathbf{C}$ if there exist double-complex function $f(\alpha)$ defined in G, such that $\omega = df(\alpha)$. Let the 1-form $\omega = \varphi(\alpha)d\alpha + \psi(a)da^*$ be exact in the domain G. Then, there exists a function $f(\alpha)$, defined in G, such that:

$$\omega = df(\alpha) = \frac{\partial f}{\partial\alpha}d\alpha + \frac{\partial f}{\partial\alpha^*}d\alpha^*.$$

Comparing, we obtain the following system for the function $f(\alpha)$:

$$\frac{\partial f}{\partial\alpha} = \varphi(\alpha), \quad \frac{\partial f}{\partial\alpha^*} = \psi(\alpha).$$

The integration of this system depends on the topological properties of the domain G.

Denoting by $H^0(G)$ the vector space of all double-complex holomorphic functions on the domain $G \subset \mathbf{C} \times \mathbf{C}$, and by $H^1(G)$ the vector space of the double-complex holomorphic 1-forms on G, we consider the sequence of mappings defined by the exterior derivative d:

$$H^0(G) \to H^1(G) \to 0.$$

This sequence is exact iff each double-complex holomorphic 1-form on G is exact 1-

form. This is true, for example, in the bi-disk $G = \Delta(z) \times \Delta(w)$, where $\Delta(z)$, resp. $\Delta(w)$, is an open disk in the complex plane $\mathbf{C}(z)$, resp. $\mathbf{C}(w)$.

2. In this paragraph we consider the equation $\partial\partial^* f(\alpha) = 0$. More precisely, the left-hand side looks as follows:

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial f}{\partial\alpha^*}\right) = \frac{1}{4}\left\{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\alpha^*}\right) - ji\frac{\partial}{\partial w}\left(\frac{\partial f}{\partial\alpha^*}\right)\right\}.$$

After some calculations we obtain:

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial f}{\partial\alpha^*}\right) = \frac{1}{4}\left(\frac{\partial^2 f}{\partial z^2} + i\frac{\partial^2 f}{\partial w^2}\right) = 0.$$

In terms of the even part $f_0(z,w)$ and the odd part $f_1(z,w)$ of the double-complex function $f(\alpha)$, the last equation reduces to the system:

$$\frac{\partial^2 f_0}{\partial z^2} + i\frac{\partial^2 f_0}{\partial w^2} = 0,$$
$$\frac{\partial^2 f_1}{\partial z^2} + i\frac{\partial^2 f_1}{\partial w^2} = 0.$$

So that, $f_0(z,w)$ and $f_1(z,w)$ are solutions of the equation

$$\frac{\partial^2 u}{\partial z^2} + i\frac{\partial^2 u}{\partial w^2} = 0 \tag{2}$$

with respect to $u = u(z,w)$ as a function of two complex variables.

3. Double-complex Laplacian. We have obtained in the previous paragraph the complex second order equation (2). It is called double-complex Laplace equation and its left-hand side – double-complex Laplace operator or double-complex Laplacian. Let us remark that this equation follows directly from the system (1) written in the previous paragraph.

In the bi-complex function theory we have the bi-complex Cauchy-Riemann system, namely

$$\frac{\partial f_0}{\partial z} = \frac{\partial f_1}{\partial w}, \quad \frac{\partial f_0}{\partial w} = -\frac{\partial f_1}{\partial z}. \tag{3}$$

From this system it follows directly the following Laplace equation

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial w^2} = 0. \tag{4}$$

It is called bi-complex Laplace equation [5], and its left-hand side – bi-complex Laplacian.

We consider the solutions of the two obtained Laplacians, the bi-complex and the double-complex Laplace equations. Their solutions are called, respectively, bi-complex harmonic functions, and double-complex harmonic functions.

It is clear, that the even and the odd parts of a double-complex function are double-complex harmonic functions. The same is true for bi-complex functions and their even and odd parts. Here we give an example of a double-complex holomorphic function which does not satisfy the bi-complex Laplace equation. Namely, let us take the function $f(\alpha) = \alpha^2$, $\alpha \in \mathbf{C}(1,j)$. In complex coordinates it looks as follows: $f(z+jw) = (z+jw)^2 = z^2 + iw^2 + j(2zw)$. The even part $f_0 = z^2 + iw^2$ and the odd part $f_1 = 2zw$ satisfy the system

(1), but they are not separately double-complex holomorphic functions.

The even part of the considered double-complex function, considered as a quadratic form $Q = z^2 + iw^2$, has a nice geometric interpretation related with the isotropic cone of a quadratic geometry over the algebra of double-complex numbers. We will see that there is not a similar interpretation for the quadratic form $Q = z^2 + w^2$.

4. Geometric application. Quadratic geometries over some algebras. Here we shall consider different quadratic forms and the corresponding isotropic cone geometry from analytical point of view. The mentioned geometry with respect to quadratic form Q is called Q-geometry [6].

4.1. The algebra of complex numbers C. The module of the complex number z is defined by the real quadratic form $x^2 + y^2 = z\overline{z} = |z|^2$. It is naturally related to the ordinary Hermitian scalar product

$$< z, \overline{w} >= z\overline{w}, \quad < z, \overline{z} >= x^2 + y^2.$$

The equation $< z, \overline{z} >= 0$ defines the isotropic cone, which is trivial: it reduces to the origin. This simple example serves only for comparison.

4.2. Q-geometry with $Q = z^2 + w^2$. We can take the complex quadratic form $z^2 + w^2$. It is related with the scalar product $< \alpha, \beta >= zu + wv$, where $z, w, u, v \in \mathbf{C}$. We have in mind the basic equalities $< 1, 1 >= 1$, $< 1, j >= < j, 1 >= 0$, $< j, j >= 1$. The isotropic cone $< \alpha, \alpha >= 0$ is defined just by the quadratic form $z^2 + w^2$. Clearly, the considered form is a holomorphic function of two complex variables which does not satisfy neither the system (3), nor the system (1). So that, in the considered Q-geometry the isotropic cone $z^2 + w^2 = 0$ cannot be interpretted from analytic point of view (bi-complex or double-complex).

In real coordinates the equation $z^2 + w^2 = 0$ is represented as follows: if $z = x + iy$, $w = \xi + i\eta$, then it is equivalent to the system

$$x^2 + \xi^2 - (y^2 + \eta^2) = 0, \quad xy + \xi\eta = 0.$$

This means that the considered isotropic cone coincides with the interesection of two 3-dimensional real surfaces in $\mathbf{R}^4$.

4.3. Q-geometry with $Q = z^2 + iw^2$. Now we consider our main problem. We take the scalar product over $\mathbf{C}(1, j)$ defined by the following basic equalities $< 1, 1 >= 1, < 1, j >=< j, 1 >= 0, < j, j >= i$. The scalar product seems as follows $< \alpha, \beta >= zu + iwv$, where $\alpha = z + jw$, and $\beta = u + jv$, $j^2 = i$, $z, w, u, v \in C$. The isoptropic cone $< \alpha, \alpha >= z^2 + iw^2 = 0$ is defined by the holomorphic function of two complex variables $z^2 + iw^2$, which is not double-complex holomorphic. It is easy to verify that this function satisfies the double-complex Laplace equation:

$$\frac{\partial^2 u}{\partial z^2} + i \frac{\partial^2 u}{\partial w^2} = 0.$$

Finaly, we can formulate the following

Proposition. *In the above defined Q-geometry over the algebra of double-complex numbers the surface of the isotropic cone is a double-complex harmonic surface. This is not true in the sense of bi-complex numbers.*

The complex equation $z^2 + iw^2 = 0$ is equivalent to the following system in real variables:

$$x^2 - y^2 - 2\xi\eta = 0, \quad \xi^2 - \eta^2 + 2xy = 0.$$

So that, the isotropic cone in this case coincides with intersection of the above defined two 3-dimensional surfaces in $\mathbf{R}^4$.

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ДВОЙНО-КОМПЛЕКСНИ ДИФЕРЕНЦИАЛНИ ФОРМИ

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Двойно-комплексната теория на функциите [1, 2, 3] е комутативна версия на кватернионния анализ подобна на предшестващата я би-комплексна теория на функциите, начената от К. Сегре (1982) [4]. В началото беше допуснато по недоразумение идентифициране с би-комплексната теория (вж. [2, 3]). Елементите на двойно-комплексната алгебра, означавана с $\mathbf{C}(1, j)$, се представят както следва: $\alpha = z + jw$, където $j^2 = i$, z, w са комплексни числа, α е по дефиниция двойно-комплексно число. Алгебрата $\mathbf{C}(1, j)$ не е алгебра с деление. Двойно-комплексната холоморфност се определя от един вид система на Коши-Риман (вж. [1, 2]). В тази бележка развиваме някои основни понятия за диференциалните форми върху $\mathbf{C}(1, j)$. Проблемът е да докажем, че повърхнината на изотропния конус в двойно-комплексната псевдо-евклидова геометрия удовлетворява двойно-комплексно $\partial\partial^*$-уравнение.