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DYNAMIC INSTABILITY OF A CANTILEVERED TIMOSHENKO BEAM UNDER TENSILE FOLLOWER FORCE*

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The dynamic stability of a cantilevered Timoshenko beam resting on an elastic foundation of Winkler type and subjected to a tensile follower force is studied. It is found that in both cases, with and without foundation, the Timoshenko beam theory predicts dynamic instability of cantilevers under tension unlike the Bernoulli-Euler beam theory.

1. Introduction. It is well known that within the framework of the Bernoulli-Euler beam theory a cantilever subjected to a tensile follower force never loses its stability. However, it turned out that this is not the case when the cantilever is treated according to Timoshenko beam theory.

The dynamic stability of a cantilevered Timoshenko beam subjected to a compressive follower force is studied by Nemat-Nasser [1]. He found that the critical force of the Timoshenko beam is less than the critical force of the Bernoulli-Euler beam and that the critical force depends on the slenderness of the beam. The effect of the Winkler foundation on the stability of the foregoing beam is studied by Lee, Kuo and Lin [2]. They found that for some values of the foundation modulus the cantilevered Timoshenko beam is destabilized in the sense that its critical force is less than the critical force of the same beam but without foundation.

There exist many other studies on the dynamic stability of Timoshenko beams that account for different effects – concentrated masses, elastically supported ends, intermediate supports, etc. In all of them the Timoshenko beam is subjected to a compressive follower force. To the best of our knowledge there do not exist papers where results concerning the dynamic stability of Timoshenko beams subjected to tensile follower force are reported. This problem is studied in the present note. The aim is to find out whether a tensile force can destabilize a Timoshenko beam and to analyse the influence of the Winkler foundation on such phenomenon.

2. Boundary-value problem. Consider an uniform elastic cantilevered beam of length L, cross-section area A, inertia moment of the cross-section I, resting on a Winkler foundation of modulus c and subjected to a tensile force N at the free end, which is always

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normal to the end cross-section (that is a follower force[1]). In this case, according to the Timoshenko beam theory [4, 5], the local kinetic and strain energies of the beam are

$$T = \frac{1}{2}\rho A\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{2}\rho I\left(\frac{\partial \Theta}{\partial t}\right)^2,$$

$$U = \frac{1}{2}EI\left(\frac{\partial \Theta}{\partial X}\right)^2 + \frac{1}{2}kGA\left(\frac{\partial W}{\partial X} - \Theta\right)^2 + \frac{1}{2}cW^2 + \frac{1}{2}N\left(\frac{\partial W}{\partial X}\right)^2,$$

where X is the axial coordinate along the beam axis, t is the time, $W(X,t)$ is the transverse deflection of the beam axis and $\Theta(X,t)$ is the rotation angle of the cross section. E, G and k are Young's modulus, the shear modulus and the shear coefficient, respectively, and ρ is the mass density per unit length of the beam.

The Euler-Lagrange equations associated with the functional of Lagrangian density $T-U$ read

$$EI\frac{\partial^2\Theta}{\partial X^2} + kAG\left(\frac{\partial W}{\partial X} - \Theta\right) = \rho I\frac{\partial^2\Theta}{\partial t^2},$$

$$kAG\left(\frac{\partial^2 W}{\partial X^2} - \frac{\partial \Theta}{\partial X}\right) + N\frac{\partial^2 W}{\partial X^2} - cW = \rho A\frac{\partial^2 W}{\partial t^2}. \tag{1}$$

These equations together with an appropriate set of boundary conditions describe entirely the dynamic behaviour of the considered beam. The boundary conditions for a cantilevered Timoshenko beam subjected to a follower force are

$$W|_{X=0} = 0, \quad \Theta|_{X=0} = 0, \quad \left.\frac{\partial \Theta}{\partial X}\right|_{X=L} = 0, \quad \left.\frac{\partial W}{\partial X} - \Theta\right|_{X=L} = 0. \tag{2}$$

Using the dimensionless variables

$$x = \frac{1}{L}X, \quad \tau = t\sqrt{\frac{EI}{L^4\rho A}}, \quad w = \frac{1}{L}W,$$

introducing the parameters

$$\beta = \frac{k}{2(1+\nu)}, \quad \lambda = \frac{I}{L^2 A}, \quad k_0 = \frac{L^2}{AE}c, \quad P = \frac{1}{AE}N,$$

where ν is Poisson's ratio, and taking into account the relation $G = E/[2(1+\nu)]$, equations (1) and boundary conditions (2) take the dimensionless form

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\beta}{\lambda}\left(\frac{\partial w}{\partial x} - \Theta\right) = \lambda\frac{\partial^2\Theta}{\partial \tau^2},$$

$$(\beta + P)\frac{\partial^2 w}{\partial x^2} - \beta\frac{\partial \Theta}{\partial x} - k_0 w = \lambda\frac{\partial^2 w}{\partial \tau^2}, \tag{3}$$

$$w|_{x=0} = 0, \quad \Theta|_{x=0} = 0, \quad \left.\frac{\partial \Theta}{\partial x}\right|_{x=1} = 0, \quad \left.\frac{\partial w}{\partial x} - \Theta\right|_{x=1} = 0. \tag{4}$$

Separating the variables in the form

$$w = u(x)\exp(i\omega\tau), \quad \Theta = \theta(x)\exp(i\omega\tau),$$

equations (3) and conditions (4) transform to the two-point boundary value problem

[1]For a detail discussion on the notion *follower force* see the exhaustive survey of Elishakoff [3]

$$\frac{d^2\theta}{dx^2} + \frac{\beta}{\lambda}\left(\frac{du}{dx} - \theta\right) + \lambda\omega^2\theta = 0,$$

$$\frac{d^2u}{dx^2} - \frac{\beta}{\beta + P}\frac{d\theta}{dx} + \frac{\lambda\omega^2 - k_0}{\beta + P}u = 0, \tag{5}$$

$$u|_{x=0} = 0, \quad \theta|_{x=0} = 0, \quad \left.\frac{d\theta}{dx}\right|_{x=1} = 0, \quad \left.\frac{du}{dx} - \theta\right|_{x=1} = 0. \tag{6}$$

Actually, this constitutes a non-self-adjoint eigenvalue problem, the eigenvalue parameter being the frequency ω.

The general solution of the equations (5) can be written in the form

$$u = C_1 \cosh(a_1 x) - C_2 \sinh(a_2 x) + C_3 \sinh(a_1 x) + C_4 \cosh(a_2 x),$$

$$\theta = C_1 b_1 \sinh(a_1 x) + C_2 b_2 \cosh(a_2 x) + C_3 b_1 \cosh(a_1 x) - C_4 b_2 \sinh(a_2 x), \tag{7}$$

where C_i $(i = 1, \ldots, 4)$ are arbitrary complex numbers,

$$a_1 = \frac{1}{2}\sqrt{-2a - 2\sqrt{a^2 - 4b}}, \quad a_2 = \frac{1}{2}\sqrt{-2a + 2\sqrt{a^2 - 4b}},$$

$$b_1 = \left(a_1^2 + \gamma\right)\frac{\beta + P}{\beta a_1}, \quad b_2 = -\left(a_2^2 + \gamma\right)\frac{\beta + P}{\beta a_2},$$

$$\gamma = \frac{\lambda\omega^2 - k_0}{\beta + P}, \quad a = \gamma - \frac{\beta P}{\lambda(\beta + P)} + \lambda\omega^2, \quad b = \gamma\left(\lambda\omega^2 - \frac{\beta}{\lambda}\right).$$

Substituting the solution (7) in the boundary conditions (6), one obtains a linear homogeneous system for the unknown constants C_i. The condition for existence of a nontrivial solution to this system can be written as

$$\begin{aligned}&\left[b_2^2 a_2 (b_1 - a_1) - b_1^2 a_1 (a_2 + b_2)\right]\cosh a_1 \cosh a_2\\ &+ \left[b_1^2 a_2 b_2 - a_1 b_1 b_2 (2a_2 + b_2)\right]\sinh a_1 \sinh a_2\\ &- b_1 b_2\left(a_1^2 + a_2^2 + b_2 a_2 - b_1 a_1\right) = 0.\end{aligned} \tag{8}$$

Consequently, for a given set of the beam parameters λ, β, k_0 and P, the eigenfrequencies ω are determined as the solutions of equation (8). The critical force P_{cr} is determined as the lowest value of P at which this equation has a solution with negative imaginary part, the rest of the beam parameters being kept fixed.

3. Results and discussion. In the case studies presented below, Timoshenko beams of rectangular cross section with shear coefficient $k = 5/6$ and Poisson's ratio $\nu = 0.3$ are considered. The eigenfrequencies of the beams are determined solving equation (8) numerically using the routine *FindRoot* in *Matematica*. Each eigenfrequency of negative imaginary part is also verified by a *Maple* implementation of the shooting method (package *shoot*[2] [6]) to confirm once again that it corresponds to a nonzero solution (u, θ) of the eigenvalue problem (5), (6).

First, the dynamic stability of beams without foundation is studied. For convenience, a new slenderness parameter μ is introduced which, for beams of rectangular cross section of height h, is $\mu = h/L$ $(\lambda = \mu^2/12)$. The critical forces are computed for values of the parameter μ between 0.01 and 0.20. As a typical example, the results for the case $\mu = 0.10$

[2]This package can be downloaded from the website of the first author of paper [6] (Douglas B. Meade) at `http://www.math.sc.edu/~meade/maple/Shoot9/Shoot9.zip`

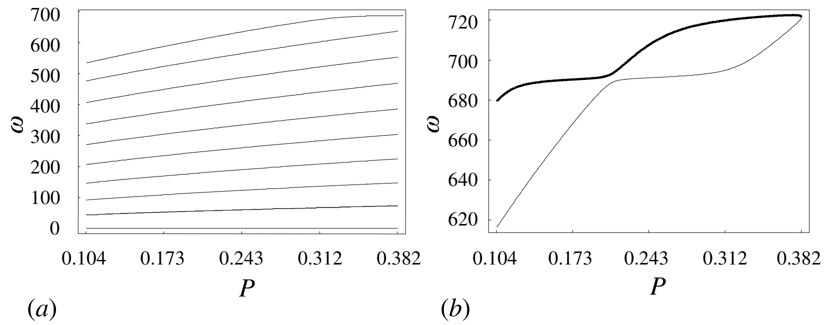

Fig. 1. Evolution of the eigenfrequencies with the force P of a Timoshenko beam with $k_0 = 0$ and $\mu = 0.10$: (a) first ten eigenfrequencies; (b) eleventh (thin line) and twelfth (thick line) eigenfrequencies

are presented in Fig. 1. The evolution of the first 10 eigenfrequencies is shown in Fig. 1(a). It is found that the first eigenfrequency decreases, while the eigenfrequencies from second to tenth increase with the tensile force P. The eleventh and twelfth eigenfrequencies also increase, but they coincide at the critical force $P_{cr} = 0.3817$ as is seen in Fig. 1(b). Beyond P_{cr} a complex eigenfrequency of real part 721 and a negative imaginary part appears indicating the dynamic instability of the beam. Timoshenko beams with other values of μ possess stability features that are similar to the case $\mu = 0.10$. The critical forces and the real parts of the respective eigenfrequencies for 10 values of the slenderness parameter μ are presented in Table 1.

Table 1. Critical forces for cantilevered Timoshenko beams without foundation

μ	0.01	0.02	0.03	0.04	0.05	0.10	0.15	0.20
P_{cr}	0.5705	0.5709	0.4101	0.3472	0.4095	0.3817	0.3541	0.4131
$\mathbf{Re}\,[\omega]$	67945	16992	7558	4257	2759	721	312	208

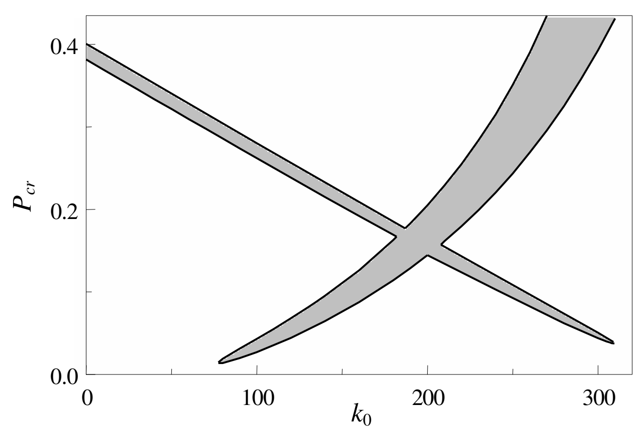

Fig. 2. Instability region (shaded) of a cantilevered Timoshenko beam with $\mu = 0.10$.

Next, the dynamic stability of Timoshenko beam of slenderness $\mu = 0.10$ resting on Winkler foundation is studied for values of the dimensionless foundation modulus k_0 up to 310. It is found that there exists a region of parameters (k_0, P), where the beam is unstable which is shown in Fig. 2. The lowest critical force is found to be $P_{cr} = 0.0136$ achieved at foundation of modulus $k_0 = 77.5$. This critical force is smaller than the compressive critical force for the same beam without foundation which is 0.0153.

Thus, it is found that the Timoshenko beam theory based on equations (1) predicts dynamic instability of cantilevers under tension unlike the Bernoulli-Euler theory. The Winkler foundation is found to destabilize the beam for values of k_0 up to 310 in the sense that the critical forces for $0 < k_0 < 310$ are less than the critical force at $k_0 = 0$.

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ДИНАМИЧНА НЕУСТОЙЧИВОСТ НА КОНЗОЛНА ГРЕДА НА ТИМОШЕНКО ПОДЛОЖЕНА НА ОСОВ ОПЪН

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Представено е изследване на динамичната устойчивост на конзолна греда на Тимошенко върху еластична основа от Винклеров тип под действието на осова сила на опън. Показано е, че както с основа, така и без нея, моделирането на тази греда посредством теорията на Тимошенко води до динамична неустойчивост, за разлика от моделирането посредством теорията на Бернули-Ойлер.