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QUASI-STATIC MOTION OF THE CONTACT LINE IN THE WILHELMY-PLATE GEOMETRY: CONTACT LINE DISSIPATION MODEL*

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We study numerically the motion of contact lines in the context of a dip coating process: a vertical solid plate is withdrawn at constant velocity from a tank of liquid. We apply the contact line dissipation model to describe the quasi-static motion of the contact line. Our study is based on the approach suggested in de Ruijter et al. [3] which uses the standard mechanical description of dissipative system dynamics and which introduces a phenomenological dissipation term proportional to the contact line length. We obtain numerically the motion of the contact line and the dependence of its dynamics on the solid plate velocity in the case of a homogeneous solid plate. We find that the time dependence of the height of the contact line is well approximated by an exponential decay function. We show how one can obtain estimates for the friction coefficient from the experimental data. The numerically obtained data is compared to the experimental results in Delon G. et al. [8] and a very good agreement is found for velocities below the entrainment transition.

1. Introduction. The spreading of a liquid on a solid and the motion of contact lines are important in many industrial processes. That is why, they are subject of numerous experimental and theoretical studies [1, 2]. It is important to achieve a certain degree of control over the spreading process and that involves the motion of the contact line too. This problem is especially challenging in the case of partial wetting regime. A number of different approaches are suggested in the literature (see, e.g., [3] and the references therein). The testing of the theoretical approaches modeling the dynamic phenomena in close vicinity of the contact line against the experimental data and determination of the range of validity of the approximations made is indispensable part of the research process. The validation of these theoretical approaches is their ability to reproduce correctly the existing experimental data.

Our goal here is to test a contact line dissipation model [3–7] with respect to experimental data [8] on the receding motion of the contact line in the Wilhelmy-plate geometry when the motion can be considered to be quasistatic. We will also compare these results with corresponding results obtained by the hydrodynamic approach within the context of lubrication theory [8, 9].

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2. Formulation. The contact line dissipation model is based on the approach suggested in [3] which uses the standard mechanical description of dissipative system dynamics and which introduces a phenomenological dissipation term T proportional to the contact line length l, i.e., $T = \int \xi \, v^2/2 \, \partial l$, v is the contact line velocity, ξ is a phenomenological friction coefficient. It is assumed here that the energy dissipated at the contact line is much larger than the viscous dissipation in the bulk of the liquid. This approach was further developed in [5–7]. In [6] for arbitrary contact line shape it is shown rigorously starting from first principles that the introduction of a friction dissipation leads to the well known equation relating the contact line velocity v and the dynamic contact angle θ [4]:

$$v = \frac{\gamma}{\xi}(\cos\theta_{eq} - \cos\theta), \tag{1}$$

where γ is the liquid surface tension, θ_{eq} is the equilibrium contact angle.

More specifically, we apply here the contact line dissipation model to the dip coating process: a vertical homogeneous solid plate is withdrawn at constant velocity u from a bath of liquid (see Fig. 1 where a schematic drawing of the dip-coating process is shown).

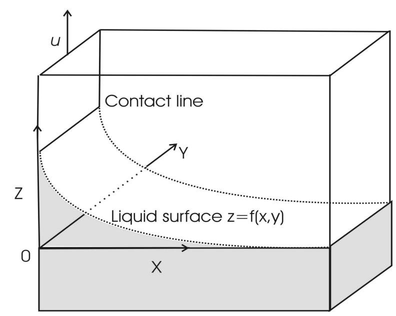

Fig. 1. Schematic drawing of the dip-coating process

In the case of a forced motion of the contact line as one has in the case of a dip coating process, eq. (1) of the contact line dissipation model is written as (see [7]):

$$\dot{h}_{cl}(t) + u = \frac{\gamma}{\xi}\left(\cos\theta_{eq} - \cos\theta\right).$$

This equation can be made dimensionless by expressing the height in terms of the capillary length l_c and the time in terms of the characteristic time $\bar{\tau}_0 = l_c \xi/\gamma$, i.e. $\bar{t} = t/\bar{\tau}_0$, and therefore the dimensionless velocity is $\bar{u} = u\,\xi/\gamma$,

$$\dot{\bar{h}}_{cl}(\bar{t}) + \bar{u} = \cos\theta_{eq} - \cos\theta \tag{2}$$

Since we consider the motion of the meniscus to be quasistatic, we can use the following relation, known from the statics, between the height of the contact line and the dynamic contact angle:

$$\bar{h}_{cl} = \sqrt{2(1 - \sin\theta)}. \tag{3}$$

From eqs. (2) and (3) one can eliminate the unknown dynamic contact angle θ and obtain the following differential equation for the height of the contact line:

$$\dot{\bar{h}}_{cl} = -\bar{u} + \cos\theta_{eq} - \bar{h}_{cl}\sqrt{1 - \bar{h}_{cl}^2/4} \tag{4}$$

One needs to know the parameters of the model, ξ and θ_{eq}, and also the initial height $\bar{h}_{cl}(0)$ of the contact line in order to determine uniquely the dependence of the height of the contact line $\bar{h}_{cl}(\bar{t})$ on the dimensionless time.

We study eq. (4) numerically by applying a standard Runge-Cutta algorithm. We find that for the studied range of values of $\bar{u}$, θ_{eq}, and initial values of the contact line height $\bar{h}_{cl}(0)$ which are greater (up to 20%) than the corresponding stationary value $\bar{h}_s(\bar{u})$, the numerically obtained time dependence of the height $\bar{h}_{cl}(\bar{t})$ is well approximated by an exponential decay function. This exponential function depends on the plate velocity and also depends weakly on the initial height $\bar{h}_{cl}(0)$ of the contact line. This result shows that this quasistatic contact line dissipation model describes well the type of the relaxation of the contact line found in the experiment [8].

In order to determine how well this model reproduces quantatively the dependence of the height and also of the relaxation time on the plate velocity, we need to "gauge" the model, i.e., we need first to determine the value of the friction coefficient ξ from the experimental data. Since we wish to compare the numerical results obtained in the framework of the contact line dissipation model with those in ref. [8], we use these experimental data to determine ξ.

The liquid used in the experiment [8] is polydimethylsiloxane (PDMS) with dynamic viscosity $\eta = 4.95$ [Ps $\cdot$ s], surface tension $\gamma = 20.3$ [mN / m], and density $\rho = 970$[kg/m^3] The capillary length $l_c = \sqrt{\gamma/\rho g}$ is 1.46 [mm] for PDMS, g is the gravity acceleration. The PDMS has a static receding contact angle $\theta_{eq}^r = 51.5°$at the plate. The velocities u in the experiment are small and are below the threshold velocity. The corresponding (dimensionless) capillary numbers $Ca = u\eta/\gamma$ are much smaller than unity ($\leq 10^{-2}$). This implies that the motion of the meniscus can be assumed to be quasistatic. It is found that the receding motion of the height $h_{cl}(t)$ of the contact line towards the stationary value $h_s(Ca)$ at fixed pull-up velocity Ca of the plate is exponential with a characteristic relaxation time $\tau(Ca)$ depending on the velocity of the plate. The dependence of the contact line stationary height on the velocity as well as the dependence of the relaxation time on the velocity were obtained below the entrainment transition and are shown in Figs. 2 and 3, correspondingly, with solid squares. In these figures the variables are made dimensionless according to the following settings: $\bar{h} = h_{cl}(Ca)/l_c$ – the height is measured in capillary lengths, the relaxation time $\tau^* = \tau\,\eta l_c/\gamma$ is measured in terms of the characteristic time $\tau_0 = \gamma/\eta l_c$ (or, correspondingly, the dimensionless relaxation rate $\sigma^* = 1/\tau^*$ is used) and the dimensionless velocity is given by the capillary number Ca.

The determination of the friction coefficient ξ can be done by comparing the numerically obtained value for h_s (or, respectively, the relaxation time τ) with the experimental one for one specific value of the plate velocity. This is done quite similarly to the determination of the slip length l_s in the hydrodynamic model [8, 9]. Of course, one can do that for all values of the velocity used in the experiment and then one can find the average of the obtained values for the friction coefficient to decrease the error.

1. For given experimental values of the dimensionless plate velocity Ca_0 and stationary

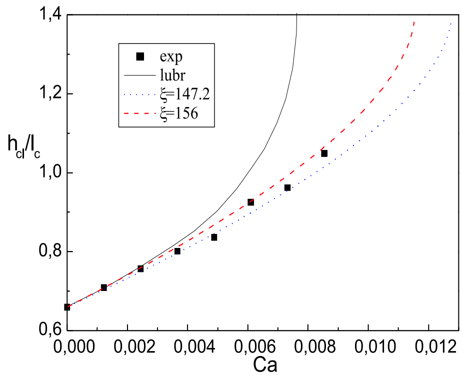

Fig. 2. The stationary height h_{cl}/l_c (measured in capillary lengths) as function of the dimensionless plate velocity Ca: the experimental results – solid squares; the lubrication approximation results – solid line; the contact line dissipation model results – dashed line for $\xi = 156$ and dotted line for $\xi = 147.2$ (ξ is in $\left[\mathrm{N \cdot s/m^2}\right]$)

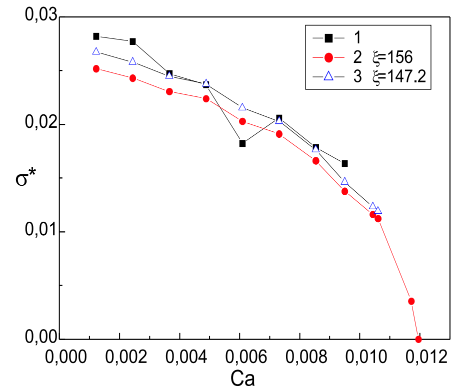

Fig. 3. The dimensionless relaxation rates σ^* ($\sigma^* = 1/\tau^*$) as function of the dimensionless plate velocity Ca: the experimental results – solid squares-solid line; the contact line dissipation model results – solid circles-solid line for $\xi = 156$ and empty triangles – solid line for $\xi = 147.2$ (ξ is in $\left[\mathrm{N \cdot s/m^2}\right]$)

height of the contact line $h_s^{\exp}(Ca_0)$, from eqs. (4) by setting $\dot{\bar{h}}_{cl} = 0$, we obtain:

$$\xi = \frac{\eta}{Ca_0}\left(\cos\theta_{eq} - h_s^{\exp}(Ca_0)/l_s\sqrt{1 - \left[h_s^{\exp}(Ca_0)/l_c\right]^2 \Big/ 4}\right)$$

For example, for the dimensionless plate velocity $Ca = 0.008534$ we find $\xi = 157.5$ $\left[\mathrm{N \cdot s/m^2}\right]$. The averaged over all plate velocities in Fig. 2 value of the friction coefficient is $\xi = 156$ $\left[\mathrm{N \cdot s/m^2}\right]$. Therefore, we get that the friction coefficient is approximately 30 times greater than the dynamic viscosity η.

2. In the second case for given from the experiment values of the plate velocity Ca and relaxation time $\tau^*(Ca_0)$ the determination of the friction coefficient ξ can be accomplished by solving a sequence of problems. That is, one finds a sequence of solutions of the eq. (4) for suitably constructed sequence of values for the friction coefficient $\{\xi_n\}$. Then, one finds the corresponding relaxation times by fitting the numerical solutions with exponential decay function. The sequence of values of the friction coefficient is such that the corresponding sequence of relaxation times$\{\tau^*(\xi_n)\}$ approaches the experimental value $\tau^*(Ca_0)$. The obtained in this way values for the friction coefficient differ slightly from the ones obtained in the first way. For example, one gets $\xi = 147.2$ $\left[\mathrm{N \cdot s/m^2}\right]$ for dimensionless speed $Ca = 0.003657$.

After ξ is determined, one can proceed to obtain from eq. 4 the stationary values of the receding height of the contact line $\bar{h}_s(Ca)$ for different pull-up velocities Ca of the moving plate and the corresponding relaxation rates σ^* ($\sigma^* = 1/\tau^*(Ca)$) for the same values of the parameters γ, η, ρ, l_c, θ_{eq} as the ones used in the experiment [8]. The obtained numerical results are shown in Fig. 2 and Fig. 3, respectively. The results for the stationary heights (Fig. 2) are shown for two values of the friction coefficient: for $\xi = 147.2\ \left[\mathrm{N}\cdot\mathrm{s/m}^2\right]$ with dotted line, and for $\xi = 156\ \left[\mathrm{N}\cdot\mathrm{s/m}^2\right]$ with dashed line. When the friction coefficient is varied between these two values, the corresponding stationary heights vary between the corresponding ones in Fig. 2. Similarly, the dimensionless relaxation rates σ^* as function of the dimensionless plate velocity Ca are shown in Fig. 3. for the same two values of the friction coefficient.

3. Conclusion. The obtained numerical data in the framework of the contact line dissipation model compare very well with the experimental results in [8]. We find in agreement with the experimental results exponential relaxation of the height of the contact line with time. Also, not only the correct types of dependences of the stationary height on the plate velocity and of the relaxation time on the plate velocity are reproduced, but, moreover, the theoretical results agree quantatively very well with the experimental (for velocities below the entrainment transition) as it can be seen from Figs. 2 and 3. The employed model gives much better results than the hydrodynamic lubrication model [8,9] (see the solid line in Fig. 2). Furthermore, this model gives an effective way of determining the phenomenologically introduced friction coefficient from the experiments in which the motion of the contact line can be considered quasistatic.

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КВАЗИСТАТИЧНО ДВИЖЕНИЕ НА КОНТАКТНАТА ЛИНИЯ ПРИ ИЗВАЖДАНЕ НА ПЛАСТИНА ОТ СЪД С ТЕЧНОСТ: МОДЕЛ НА ДИСИПАЦИЯ НА КОНТАКТНАТА ЛИНИЯ

Станимир Д. Илиев, Нина Хр. Пешева, Вадим С. Николаев

Числено се изследва движението на контактната линия при изваждане с постоянна скорост на вертикална пластина от съд с течност. Прилага се модела на дисипация на енергията в околност на контактната линия, за да се опише квазистатичното движение на контактната линия. Това изследване се базира на подхода, предложен в de Ruijter et al. [3]. В този подход се използва общоприетото в механиката описание на динамиката на дисипативни системи и се въвежда феноменологичен дисипативен член пропорционален на дължината на контактната линия. Числено се получава движението на контактната линия и зависимостта на динамиката на контактната линия от скоростта на изваждане на хомогенна пластина. Получено е, че зависимостта на височината на контактната линия от времето се описва добре от експоненциално спадаща функция. Показано е как може да се получи оценка за феноменологично въведения коефициент на дисипация от експериментални данни. Числено получените резултати в рамките на разглеждания модел са сравнени с експериментално получени резултати в Delon G. et al. [8] и е получено много добро съгласуване с експеримента.