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EXISTENCE OF SOLUTIONS IN A CLASS OF DYNAMIC SPATIAL CONSUMPTION MODELS*

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We investigate the existence of solutions to a class of economic models that feature a dynamically optimizing consumer who makes both consumption and spatial location (migration) decisions. We show that under certain assumptions a solution to the model exists.

Introduction. In the present paper we study the existence of solutions to a class of dynamic economic models featuring a combination of consumption-saving and spatial location decisions made by an optimizing agent. The problem stated here constitutes a natural building block for developing larger models to study aggregate consumption and migration phenomena. We translate the economic problem into an optimal control problem with a finite horizon and prove the existence of a solution to this problem under certain assumptions. Unlike traditional existence proofs in the spirit of Theorem 4, §4.2 in [4], we dispense with convexity assumptions on the set of generalized speeds for the optimal control problem. The setup and the assumptions we employ, are described in Section 2. Our existence result, along with its proof, is stated in Section 3.

2. The problem. We consider the following problem:

$$\max_{c(t),z(t)} J(c(t),z(t)) := \int_0^T [u(c(t)) - g(\dot{x}(t))]e^{-\rho t}dt + l(a(T)) \tag{1}$$

subject to

$$\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - h(\dot{x}(t)),\ a(0) = a_0 \geq 0, \tag{2}$$

$$\dot{x}(t) = z(t),\ x(0) = x_0 \in [0,1]. \tag{3}$$

Problem (1)–(3) is a continuous-time optimal control problem with a finite horizon T, where $a(t)$ and $x(t)$ are the state variables and $c(t)$ and $z(t)$ are the controls. The assumptions we make, are as follows. For the controls they are:

$$0 \leq c(t) \leq C,\ \forall t \in [0,T], \tag{4}$$

$$|z(t)| \leq Z,\ \forall t \in [0,T], \tag{5}$$

with $Z, C > 0$.

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We impose the nonnegativity constraint

$$a(T) \geq 0. \tag{6}$$

Then, the set of admissible controls is taken to be

$$\Delta := \{c(t) \in L_2[0,T], z(t) \in L_2[0,T], \text{ such that conditions } (4), (5) \text{ and } (6) \text{ are satisfied}\}.$$

The real function $w(x)$ is defined and continuous on the real line. We assume that $\underline{W} \leq w(x) \leq \overline{W}$ and $w(x) \geq 0$ for $x \in [0,1]$, $w(x) < 0$ for $x \notin [0,1]$.

The real-valued functions $u(\cdot)$ and $l(\cdot)$, defined on $[0,\infty)$, are increasing, continuous and concave.

The real-valued functions $g(\cdot)$ and $h(\cdot)$ defined on $[-Z,Z]$, are continuous and convex. We also assume that $g(z), h(z) \geq 0$ and that $g(0) = h(0) = 0$.

The parameters ρ, r and p are taken to be strictly positive.

In economic terms, the above model looks at the optimal decisions of an economic agent who tries to maximize utility $u(\cdot)$ from consumption $c(t)$, where a unit of consumption is obtained at price p. This consumer is characterized, at each time instant t, by financial asset holdings $a(t)$ and location in space $x(t)$. Assets earn interest r and in location x the agent can receive[1] wage $w(x)$. Apart from $c(t)$, the consumer chooses a location in space through the speed of relocation $z(t)$. Changing one's location incurs certain monetary costs (transportation, transaction etc.), denoted by $h(\cdot)$, which depend on the speed of relocation. The consumer is also assumed to form habits with respect to the current location and, thus, to experience a certain psychological discomfort (disutility) $g(\cdot)$ from relocation. Both the utility of consumption and the disutility from relocation are time-discounted at rate ρ. Finally, the consumer also derives utility $l(\cdot)$ from the level of financial assets $a(T)$ left as a bequest at the end of his lifetime.

3. The main theorem. To prove the existence of a solution to the model, we start by stating two supplementary results.

Lemma 1. *Let the functions x_i, $i = 1, 2, \ldots$, and $\bar{x}$ be defined on $[0,T]$ and let take values in the interval $[a,b]$. Let x_i tend uniformly to $\bar{x}$ as $i \to \infty$ (denoted by $x_i \rightrightarrows \bar{x}$) and $w \in C^0[a,b]$. Then, we have*

i) $\displaystyle \frac{1}{m}\sum_{i=1}^{m} x_i \rightrightarrows \bar{x}$,

ii) $w(x_m) \rightrightarrows w(\bar{x})$,

iii) $\displaystyle w\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) \rightrightarrows w(\bar{x})$.

in $[0,T]$, as $m \to \infty$.

Proof. The proof directly replicates the standard proofs of counterpart results on numerical sequences. □

Lemma 2 (The Banach-Saks Theorem)**.** *Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of elements in a Hilbert space H which are bounded in norm: $\|v_n\| \leq K = \text{const}$, $\forall n \in \mathbb{N}$. Then, there exist a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and an element $v \in H$ such that*

[1] We do not model labour-leisure decisions in this setup, so the consumer may be taken as supplying inelastically a unit of labour at each moment in time.

$$\left\| \frac{v_{n_1} + \cdots + v_{n_s}}{s} - v \right\| \to 0 \ \textit{as } s \to \infty.$$

Proof. See, for example, [1, pp. 78–81]. □

Theorem 1. *Under the assumptions stated in section* 2*, there exists a solution to problem* (1)–(3) *for* $(c(t), z(t)) \in \Delta$.

Proof. We start by noting that the set of admissible controls Δ is nonempty. To see this, choose controls $c(t) \equiv c_0 = const$ and $z(t) \equiv 0$. Then, any $c_0 \in [0, \min\{w(x_0)/p, C\}]$ ensure that $a(T) \geq 0$.

Next, observe that

$$\left| \int_0^T u(c(t)) e^{-\rho t} dt \right| \leq \int_0^T |u(c(t))| e^{-\rho t} dt \leq \int_0^T \max_{0 \leq c \leq C} |u(c)| e^{-\rho t} dt \leq T \max_{0 \leq c \leq C} |u(c)| < \infty.$$

Also,

$$0 \leq \int_0^T g(z(t)) e^{-\rho t} dt \leq \int_0^T \max_{z \leq |Z|} g(z) dt = T \max_{z \leq |Z|} g(z) < \infty.$$

Finally,

$$a(T) \leq e^{rT} [a_0 + T \max_x |w(x)|], \text{ i.e. } 0 \leq a(T) \leq \text{const} < \infty,$$

so that $l(a(T))$ is bounded.

Thus, for $(c(t), z(t)) \in \Delta$, the objective functional (1) is bounded. Consequently, $J_0 := \sup_{(c(t),z(t)) \in \Delta} J(c(t), z(t)) < \infty$. Then, we can choose a sequence of controls $\{(c_k(t), z_k(t))\} \subset \Delta$ such that $J(c_k(t), z_k(t)) \to J_0$.

Let $a_k(t)$ and $x_k(t)$ be the state variables corresponding to the controls $(c_k(t), z_k(t))$. It is easy to verify that the functions $a_k(t)$ and $x_k(t)$ form a uniformly bounded and equicontinuous set. Then, by the Arzelà-Ascoli theorem (see, e.g., [4], Ch.4), there exists a subsequence $(a_{k_s}(t), x_{k_s}(t)) \rightrightarrows (\bar{a}(t), \bar{x}(t))$.

Notice that for $(c(t), z(t)) \in \Delta$, we have

$$\|c(t)\|_{L_2} = \int_0^T |c(t)|^2 dt = \int_0^T c(t)^2 dt \leq \int_0^T C^2 dt = C^2 T,$$

$$\|z(t)\|_{L_2} = \int_0^T |z(t)|^2 dt \leq \int_0^T Z^2 dt = Z^2 T.$$

Then, if $c_{k_s}(t)$ and $z_{k_s}(t)$ are the controls corresponding to $(a_{k_s}(t), x_{k_s}(t))$, by Lemma 2 we can in turn choose subsequences $c_{k_{s_q}}(t)$ and $z_{k_{s_q}}(t)$ whose arithmetic means tend in $L_2[0,T]$-norm to some elements in $L_2[0,T]$, denoted $\bar{\bar{c}}(t)$ and $\bar{\bar{z}}(t)$, respectively. For brevity, we introduce the notation $c_q(t) := c_{k_{s_q}}(t)$, $z_q(t) := z_{k_{s_q}}(t)$ etc., as well as $\tilde{c}_m(t) := \frac{1}{m} \sum_{q=1}^{m} c_q(t)$ and $\tilde{z}_m(t) := \frac{1}{m} \sum_{q=1}^{m} z_q(t)$.

Then, we have established that: (1) $(a_q(t), x_q(t)) \rightrightarrows (\bar{a}(t), \bar{x}(t))$ as $q \to \infty$ and (2) $\tilde{c}_m(t) \xrightarrow[L_2]{} \bar{\bar{c}}(t)$, $\tilde{z}_m(t) \xrightarrow[L_2]{} \bar{\bar{z}}(t)$ as $m \to \infty$.

So far, it is not clear whether $\tilde{c}_m(t)$ and $\tilde{z}_m(t)$ are admissible: they obviously satisfy

(4) and (5) but the corresponding $a(T)$ may fail to satisfy (6). However, we can show that the controls $\bar{\bar{c}}(t)$ and $\bar{\bar{z}}(t)$ are admissible.

Indeed, according to [3, Ch. 7, §2.5, Prop. 4] we can choose subsequences of $\tilde{c}_m(t)$ and $\tilde{z}_m(t)$ that converge a.e. and, after passing to the limit, we obtain that $\bar{\bar{c}}(t)$ and $\bar{\bar{z}}(t)$ satisfy (4) and (5).

It remains to show that $\bar{\bar{a}}(T) = e^{rT}\left[a_0 + \int_0^T [w(\bar{\bar{x}}(t)) - p\bar{\bar{c}}(t) - h(\bar{\bar{z}}(t))]e^{-rt}dt\right] \geq 0$, where $\bar{\bar{x}}(t) = x_0 + \int_0^t \bar{\bar{z}}(\tau)d\tau$.

Consider

$$\tilde{a}_m(T) = e^{rT}\left[a_0 + \int_0^T [w(\tilde{x}_m(t)) - p\tilde{c}_m(t) - h(\tilde{z}_m(t))]e^{-rt}dt\right], \tag{7}$$

with $\tilde{x}_m(t) = x_0 + \int_0^t \tilde{z}_m(\tau)d\tau = \frac{1}{m}\sum_{q=1}^m \left(x_0 + \int_0^t z_q(\tau)d\tau\right) = \frac{1}{m}\sum_{q=1}^m x_q(t)$. Adding and subtracting $\frac{1}{m}\sum_{q=1}^m w(x_q(t))$, and applying Jensen's inequality to the term $h(\tilde{z}_m(t))$, we obtain

$$\begin{aligned}
\tilde{a}_m(T) \geq & e^{rT}\int_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m}\sum_{q=1}^m w(x_q(t))\right] e^{-rt}dt \\
& + \frac{1}{m}\sum_{q=1}^m e^{rT}\left[a_0 + \int_0^T [w(x_q(t)) - pc_q(t) - h(z_q(t))]e^{-rt}dt\right] \\
\geq & e^{rT}\int_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m}\sum_{q=1}^m w(x_q(t))\right] e^{-rt}dt
\end{aligned} \tag{8}$$

By Lemma 1, both integrands inside the square brackets in the last line of (8) tend uniformly to $w(\bar{x}(t))$, so that the integral tends to zero. Thus, if $\lim_{m\to\infty} \tilde{a}_m(T)$ exists, we have $\lim_{m\to\infty} \tilde{a}_m(T) \geq 0$.

We proceed to check that $\lim_{m_j\to\infty} \tilde{a}_{m_j}(T) = \bar{\bar{a}}(T)$ for a suitable subsequence $\tilde{a}_{m_j}(T)$. We know that $\frac{1}{m}\sum_{q=1}^m x_q(t) = x_0 + \int_0^t \frac{1}{m}\sum_{q=1}^m z_q(\tau)d\tau$. Since $\frac{1}{m}\sum_{q=1}^m x_q(t) \rightrightarrows \bar{x}(t)$ and, additionally, as it is easy to verify by applying Hölder's inequality that $\int_0^t \tilde{z}_m(\tau)d\tau \to \int_0^t \bar{\bar{z}}(\tau)d\tau$ when $\tilde{z}_m(t) \xrightarrow[L_2]{} \bar{\bar{z}}(t)$, we obtain $\bar{x}(t) = x_0 + \int_0^t \bar{\bar{z}}(\tau)d\tau = \bar{\bar{x}}(t)$.

Since $\tilde{c}_m(t) \xrightarrow[L_2]{} \bar{\bar{c}}(t)$ and $\tilde{z}_m(t) \xrightarrow[L_2]{} \bar{\bar{z}}(t)$, there exist subsequences $\tilde{c}_{m_j}(t)$ and $\tilde{z}_{m_j}(t)$ such that $\tilde{c}_{m_j}(t) \xrightarrow[a.e.]{} \bar{\bar{c}}(t)$ and $\tilde{z}_{m_j}(t) \xrightarrow[a.e.]{} \bar{\bar{z}}(t)$. To simplify notation, we refer to the new subsequences as $\tilde{c}_j(t)$ and $\tilde{z}_j(t)$. Since the function $h(z)$ is bounded on $[-Z, Z]$, by Lebesgue's dominated convergence theorem $\int_0^T h(\tilde{z}_j(t))e^{-rt}dt \to \int_0^T h(\bar{\bar{z}}(t))e^{-rt}dt$. It can also be verified that $\int_0^T \tilde{c}_j(t)e^{-rt}dt \to \int_0^T \bar{\bar{c}}(t)e^{-rt}dt$. Lastly, we know that $\int_0^T w(\tilde{x}_j(t))e^{-rt}dt \to \int_0^T w(\bar{x}(t))e^{-rt}dt$ as $w(\tilde{x}_j(t)) \rightrightarrows w(\bar{x}(t))$. Consequently, the limit of (7) as $m_j \to \infty$ exists and is equal to $\bar{\bar{a}}(T)$, so that $\bar{\bar{a}}(T) \geq 0$. This shows that $\bar{\bar{c}}(t)$ and $\bar{\bar{z}}(t)$ are admissible.

By an application of Lebesgue's dominated convergence theorem to the corresponding terms in (1), we get $\lim\limits_{j\to\infty} J(\tilde{c}_j(t), \tilde{z}_j(t)) = J(\bar{\bar{c}}(t), \bar{\bar{z}}(t))$.

Define $\rho_{m_j}(T) := e^{rT} \int_0^T \left[w(\tilde{x}_{m_j}(t)) - \frac{1}{m_j} \sum\limits_{q=1}^{m_j} w(x_q(t)) \right] e^{-rt} dt$. Obviously, $\tilde{\tilde{a}}_{m_j}(T) = \tilde{a}_{m_j}(T) - \rho_{m_j}(T)$ also tends to $\bar{\bar{a}}(T)$ and $\tilde{\tilde{a}}_{m_j}(T) \geq \frac{1}{m_j} \sum\limits_{q=1}^{m_j} a_q(T)$, where $a_q(T)$ corresponds to $(c_q(t), z_q(T))$. Then, indexing by j instead of m_j in order to simplify the notation, we get

$$J_0 \geq J(\bar{\bar{c}}(t), \bar{\bar{z}}(t)) = \lim_{j\to\infty} \left\{ \int_0^T [u(\tilde{c}_j(t)) - g(\tilde{z}_j(t))] e^{-\rho t} dt + l(\tilde{\tilde{a}}_j(T)) \right\}$$

$$\geq \lim_{j\to\infty} \left\{ \frac{1}{j} \sum_{i=1}^{j} \left[\int_0^T [u(c_i(t)) - g(z_i(t))] e^{-\rho t} dt + l\left(a_i(T)\right) \right] \right\}$$

$$= \lim_{j\to\infty} \left\{ \frac{1}{j} \sum_{i=1}^{j} J(c_i(t), z_i(t)) \right\} = J_0.$$

This shows that the admissible pair $(\bar{\bar{c}}(t), \bar{\bar{z}}(t))$ is optimal, as required. $\square$

4. Examples. In [2] we study the optimal behaviour of the consumer under the assumptions $u(c(t)) = c(t)^{1-\theta}/(1-\theta)$, $g(z(t)) = \eta z(t)^2$, $l(a(T)) = e^{-\rho T} a(T)^{1-\theta}/(1-\theta)$ and $h(z(t)) = \xi z(t)^2$.

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СЪЩЕСТВУВАНЕ НА РЕШЕНИЕ НА ЕДИН КЛАС ОТ ДИНАМИЧНИ ПРОСТРАНСТВЕНИ МОДЕЛИ НА ПОТРЕБЛЕНИЕ

Йордан В. Йорданов, Андрей А. Василев

В работата се разглежда въпроса за съществуване на решения за един клас от икономически модели с потребител, решаващ динамична оптимизационна задача, в която се прави избор на потреблението и на положението в пространството (миграция). Доказва се, че при определени допускания съществува решение на модела.