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CONTRACTIBILITY OF PARETO SOLUTIONS SETS IN CONCAVE VECTOR MAXIMIZATION*

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In present paper we study the Pareto solutions in concave vector maximization with compact and convex domain. One of the most important problems in this optimization is the investigation of the topological structure of the Pareto solutions sets. We consider the problem of construction of a retraction from the feasible domain onto Pareto-optimal set, if the objective functions are concave and one of them is strictly quasi-concave on feasible domain. Using this result, it is also proved that the Pareto-optimal and Pareto-front sets are homeomorphic and contractible.

1. Introduction. The basic aim of this paper is first to show how we can construct a retraction from the feasible domain onto Pareto-optimal set in concave vector maximization problem. Next, using this function we prove that the Pareto-optimal and Pareto-front sets are homeomorphic and contractible.

The general vector maximization problem is to find $x \in X \subset \mathbb{R}^m$, $m \geq 1$, so that to maximize $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$ subject to $x \in X$, where the feasible domain X is nonempty, $J = \{1, 2, \dots, n\}$ is the index set, $n \geq 2$, $f_i : X \to R$ is a given continuous objective function for all $i \in J$.

Since the objective functions $\{f_j\}_{j=1}^{n}$ may be conflicting with each other, it is usually difficult to obtain the global maximum for each objective function at the same time. Therefore, the target of the vector maximization problem is to achieve a set of solutions that are Pareto-optimal.

Definition 1. *A point $x \in X$ is called Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J$ and $f_k(y) > f_k(x)$ for some $k \in J$. The set of the Pareto-optimal solutions of X is denoted by* $\operatorname{Max}(X, F)$ *and is called Pareto-optimal set. The set $F(\operatorname{Max}(X, F))$ is called Pareto-front set.*

One of the most important problems of vector maximization is the investigation of the topological structure of the Pareto solutions sets (Pareto-optimal set and Pareto-front set). The investigation of the topological properties of Pareto solutions sets is started by [7], see also [2] and [5].

A well-known open problem in vector optimization is the contractibility of the Pareto solutions sets. The basic results are as follows:

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In [1], it is proved that the Pareto-optimal set in strictly quasi-concave vector maximization with compact and convex feasible domain is contractible.

In [4], it is proved that the Pareto-optimal and the Pareto-front sets in strictly quasi-concave vector maximization with convex domain are contractible, if any intersection of level sets of the objective functions with the feasible domain is a compact set.

The paper is organized as follows: In Section 2 some definitions and notions from topology and optimization theory are given. In Section 3 a retraction from the feasible domain onto the Pareto-optimal set is constructed and it is proved that the Pareto-optimal and Pareto-front sets are homeomorphic and contractible.

2. General definitions and notions. Let a function $dis : X \times X \to R_+$ be a metric on X. In a metric space (X, dis), let τ be the topology induced by dis. In a topological space (X, τ), for $Y \subset X$ we recall some topological definitions:

Definition 2. *The set Y is a retract of X if and only if there exists a continuous function $r : X \to Y$ such that $r(x) = x$ for all $x \in Y$ (the restriction of r is the identity function on Y). The function r is called a retraction.*

Definition 3. *A continuous function $d : X \times [0; 1] \to X$ is a deformation retraction of X onto $Y \subset X$ if and only if $d(x, 0) = x$, $d(x, 1) \in Y$, and $d(a, t) = a$ for all $x \in X$, $a \in Y$, and $t \in [0; 1]$. The set Y is called a deformation retract of X.*

From a more formal viewpoint, a retraction is a function $r : X \to Y$ such that $r \circ r(x) = r(x)$ for all $x \in X$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics. Clearly, every deformation retract is a retract, but in generally the converse does not hold [3].

Definition 4. *The set Y is contractible if and only if there exist a continuous function $c : Y \times [0; 1] \to Y$ and $a \in Y$ such that $c(x, 0) = a$ and $c(x, 1) = x$ for all $x \in Y$.*

Of course, the set Y is contractible if there exists a deformation retract of Y onto a point. In other words, any set having a deformation retract onto a point is contractible. However, the converse is false. There are examples of contractible sets which have not deformation retract onto any point.

It is known that: convexity implies contractibility, contractibility implies path-connectedness and path-connectedness implies connectedness. But, in general, the converses do not hold [3], [6].

Definition 5. *Let X and Y be topological spaces and let $h : X \to Y$ be bijective. Then, h is called a homeomorphism if and only if the functions h and h^{-1} are continuous. If such a homeomorphism h exists, then X and Y are called homeomorphic (or X is homeomorphic to Y).*

It is clear that the contractibility of sets is preserved under homeomorphisms.

In addition, we also introduce the following notations: for every two vectors $x, y \in \mathbb{R}^n$, $x(x_1, x_2, \ldots, x_n) \geq y(y_1, y_2, \ldots, y_n)$ means $x_i \geq y_i$ for all $i \in J$, $x(x_1, x_2, \ldots, x_n) > y(y_1, y_2, \ldots, y_n)$ means $x_i > y_i$ for all $i \in J$, and $x(x_1, x_2, \ldots, x_n) \succcurlyeq y(y_1, y_2, \ldots, y_n)$ means $x_i \geq y_i$ for all $i \in J$ and $x_k > y_k$ for some $k \in J$.

Concavity and quasi-concavity of functions play a central role in optimization theory. We use the definitions of concave, quasi-concave and strictly quasi-concave functions in the usual sense:

(1) A function f_i is concave on X if and only if for any $x, y \in X$ and $t \in [0;1]$, $f_i\left(tx + (1-t)\, y\right) \geq t f_i\left(x\right) + (1-t)\, f_i\left(y\right)$.

(2) A function f_i is quasi-concave on X if and only if for any $x, y \in X$ and $t \in [0;1]$, $f_i\left(tx + (1-t)\, y\right) \geq \min(f_i\left(x\right), f_i\left(y\right))$.

(3) A function f_i is strictly quasi-concave on X if and only if for any $x, y \in X$, $x \neq y$ and $t \in (0;1)$, $f_i\left(tx + (1-t)\, y\right) > \min(f_i\left(x\right), f_i\left(y\right))$.

3. Main result. Now, under our assumptions, the functions $\{f_j\}_{j=1}^n$ are concave and a function f_λ of $\{f_j\}_{j=1}^n$ is strictly quasi-concave on the compact and convex feasible domain X, we prove that the Pareto-optimal set is a retract of the feasible domain and the Pareto solutions sets are homeomorphic and contractible.

Define a function $f : X \to \mathbb{R}$ by $f\left(x\right) = \sum_{j=1}^n f_j\left(x\right)$ for all $x \in X$. It is clear that the function f is concave on X, and $\operatorname{Arg\,max}\left(f, X\right) \subset \operatorname{Max}\left(X, F\right)$.

Define also a point-to-set mapping $\psi : X \rightrightarrows X$ by $\psi\left(x\right) = \{y \in X | F(y) \geq F(x)\}$ for all $x \in X$. It can be shown that the set $\psi\left(x\right)$ is a nonempty, compact and convex set for all $x \in X$ and $x \in \psi\left(x\right)$. Hence, the point-to-set mapping ψ is compact-valued and convex-valued on X.

These notes allow to present the main theorem of our paper:

Theorem 1. (*a*) *There exists a retraction* $r : X \to \operatorname{Max}(X, F)$ *such that* $r\left(X\right) = \operatorname{Max}\left(X, F\right)$ *and* $r\left(x\right) = \operatorname{Arg\,max}\left(f, \psi(x)\right)$ *for all* $x \in X$*;* (*b*) $\operatorname{Max}(X, F)$ *and* $F(\operatorname{Max}\left(X, F\right))$ *are homeomorphic and contractible.*

Proof. We prove (a) in two steps. In the first step the retraction $r : X \to \operatorname{Max}\left(X, F\right)$ is constructed. In the second step continuity of r is shown.

First step.

Let fix $x \in X$. Consider an optimization problem with single objective function as follows: maximize $f(y)$ subject to $y \in \psi(x)$. We show that this problem has a unique solution $\bar{x} \in \operatorname{Max}\left(X, F\right)$. Thus, a retraction $\bar{x} = r(x)$ will be constructed.

Lemma 1. *If* $x \in X$*, then* $|\operatorname{Arg\,max}(f, \psi(x))| = 1$ *and* $\operatorname{Arg\,max}(f, \psi(x)) \subset \operatorname{Max}\left(X, F\right)$.

Proof. Clearly $|\operatorname{Arg\,max}(f, \psi(x))| \geq 1$. Suppose that there exist $y_1, y_2 \in \operatorname{Arg\,max}(f, \psi(x))$, $y_1 \neq y_2$, and take $t \in (0;1)$ and $z = ty_1 + (1-t)y_2$. It is clear that the set $\operatorname{Arg\,max}(f, \psi(x))$ is convex, therefore, $z \in \operatorname{Arg\,max}(f, \psi(x))$. Thus, we obtain $f(z) = f(y_1) = f(y_2)$.

On the other hand, for each $i \in J$ we have $f_i(z) \geq t f_i(y_1) + (1-t) f_i(y_2)$. By using this result we derive that $f(z) \geq t f(y_1) + (1-t) f(y_2) = f(y_1)$. Since $f(z) = f(y_1)$, we get $f_i(z) = t f_i(y_1) + (1-t) f_i(y_2)$ for all $i \in J$ and for all $t \in (0;1)$. As a result $f_i(z) = f_i(y_2) + t(f_i(y_1) - f_i(y_2))$ for all $t \in (0;1)$, therefore, we find that $f_i(y_1) = f_i(y_2)$ for all $i \in J$.

Now, fix $t \in (0;1)$. As it was described earlier, the function f_λ is strictly quasi-concave, therefore we obtain $f_\lambda(z) > \min(f_\lambda(y_1), f_\lambda(y_2)) = f_\lambda(y_1)$. But $f_i(z) \geq t f_i(y_1) + (1-t) f_i(y_2)$ for all $i \in J$ and by using this result we derive that $f(z) > t f(y_1) + (1-t) f(y_2) = f(y_1)$. This leads to a contradiction, therefore, we obtain $|\operatorname{Arg\,max}(f, \psi(x))| = 1$.

Let us choose $y \in \operatorname{Arg\,max}(f, \psi(x))$ and assume that $y \notin \operatorname{Max}(X, F)$. From the assumption $y \notin \operatorname{Max}\left(X, F\right)$ it follows that there exists $z \in X$ such that $F(z) \succeq F(y)$. As a result we have $z \in \psi(x)$ and $f(z) > f(y)$. Again, this leads to a contradiction, therefore, we obtain $y \in \operatorname{Max}\left(X, F\right)$. The lemma is proved.

Using the results of Lemma 1, we are in a position to construct a function $r : X \to \mathrm{Max}\,(X,F)$ such that $r\,(x) = \mathrm{Arg}\,\max\,(f,\psi(x))$ for all $x \in X$.

Lemma 2. *If $x \in X$, then $x \in \mathrm{Max}\ (X,F)$ is equivalent to $\{x\} = \psi(x)$.*

Proof. Let $x \in \mathrm{Max}\ (X,F)$ and assume that $\{x\} \neq \psi(x)$. From the both conditions $x \in \psi(x)$ and $\{x\} \neq \psi(x)$ it follows that there exists $y \in \psi(x)\backslash\{x\}$ such that $F(y) \succcurlyeq F(x)$. Let us choose $t \in (0;1)$ and $z = tx + (1-t)y$, then $z \in \psi(x)$. But $x \neq y$ implies $f_\lambda(z) > f_\lambda(x)$, which contradicts the condition $x \in \mathrm{Max}\,(X,F)$. Then, we obtain $\{x\} = \psi(x)$.

Conversely, let $\{x\} = \psi(x)$ and assume that $x \notin \mathrm{Max}\,(X,F)$. From the assumption $x \notin \mathrm{Max}\,(X,F)$ it follows that there exists $y \in X$ such that $F(y) \succcurlyeq F(x)$. Thus, we deduce that $y \in \psi(x)$ and $x \neq y$, which contradicts the condition $\{x\} = \psi(x)$. Then, we obtain $x \in \mathrm{Max}\,(X,F)$. The lemma is proved.

Now, by applying the previous lemma we have that if $x \in \mathrm{Max}\ (X,F)$, then $x = r(x)$ and if $x \in X\backslash\mathrm{Max}\ (X,F)$, then $x \neq r(x)$. It is easily verified that $r \circ r = r$.

Lemma 3. $r\,(X) = \mathrm{Max}\ (X,F)$.

Proof. From Lemma 1 it follows that $r\,(X) \subset \mathrm{Max}\,(X,F)$. Applying Lemma 2, we deduce $r\,(\mathrm{Max}\,(X,F)) = \mathrm{Max}\,(X,F)$. This means that $r\,(X) = \mathrm{Max}\,(X,F)$. The lemma is proved.

Lemmas 1, 2 and 3 have shown that the function r fixes every point of $\mathrm{Max}\,(X,F)$.

Second step.

We will analyze the point-to-set mapping ψ. Using the Maximum Theorem, one of the fundamental results of optimization theory, we will show that the function r is continuous.

Lemma 4. *The point-to-set mapping ψ is continuous on X.*

Proof. First, we prove that if $\{x_k\}_{k=1}^{\infty}$,$\{y_k\}_{k=1}^{\infty} \subset X$ is a pair of sequences such that $\lim\limits_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \psi\,(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\psi\,(x_0)$.

The assumption $y_k \in \psi\,(x_k)$ for all $k \in \mathbb{N}$ implies $F\,(y_k) \geq F\,(x_k)$ for all $k \in \mathbb{N}$. From the condition $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent subsequence $\{q_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} q_k = y_0 \in X$. Therefore, there exists a convergent subsequence $\{p_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $q_k \in \psi\,(p_k)$ and $\lim\limits_{k\to\infty} p_k = x_0$. Thus, we find that $F\,(q_k) \geq F\,(p_k)$ for all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$, we obtain $F\,(y_0) \geq F\,(x_0)$. This implies $y_0 \in \psi\,(x_0)$. This means that the point-to-set mapping ψ is upper semi-continuous on X [6].

Second, we prove that if $\{x_k\}_{k=1}^{\infty} \subset X$ is a convergent sequence to $x_0 \in X$ and $y_0 \in \psi\,(x_0)$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi\,(x_k)$ for all $k \in N$ and $\lim\limits_{k\to\infty} y_k = y_0$.

As usually, let us denote the distance between the point $y_0 \in X$ and the set $\psi\,(x_k) \subset X$ by $d_k = \inf\{dis\,(y_0,x)\,|x \in \psi\,(x_k)\}$. As it was described earlier, $\psi\,(x_k)$ is a nonempty, compact and convex set, therefore, if $y_0 \notin \psi\,(x_k)$, then there exists a unique $\bar{y} \in \psi\,(x_k)$ such that $d_k = d\,(\bar{y},y_k)$.

There are two cases as follows: if $y_0 \in \psi\,(x_k)$, then $d_k = 0$ and let $y_k = y_0$; if $y_0 \notin \psi\,(x_k)$, then $d_k > 0$ and let $y_k = \bar{y}$.

So that, we get a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi\,(x_k)$ for all $k \in N$ and $d_k = dis\,(y_0,y_k)$. Since $\lim\limits_{k\to\infty} x_k = x_0$, the sequence

$\{d_k\}_{k=1}^{\infty}$ is convergent and $\lim\limits_{k\to\infty} d_k = 0$. Finally, we obtain $\lim\limits_{k\to\infty} y_k = y_0$. This means that the point-to-set mapping ψ is lower semi-continuous on X [6]. Hence, the point-to-set mapping ψ is continuous on X. The lemma is proved.

Lemma 5 [8, Theorem 9.14 – The Maximum Theorem]**.** *Let $S \subset R^n$, $\Theta \subset R^m$, $g : S \times \Theta \to R$ be a continuous function, and let $D : \Theta \rightrightarrows S$ be a compact-valued and continuous point-to-set mapping. Then, the function $g* : \Theta \to R$ defined by $g*(\theta) = \max\{g(x,\theta)\,|x \in D(\theta)\}$ is continuous on Θ, and the point-to-set mapping $D* : \Theta \rightrightarrows S$ defined by $D*(\theta) = \{x \in D(\theta)\,|g(x,\theta) = g*(\theta)\}$ is compact-valued and upper semi-continuous on Θ.*

Lemma 6. *The function r is continuous on X.*

Proof. Applying Lemma 5 for $S = \Theta = X$ and $\psi = D$, we derive that the function f is continuous on X. As it was mentioned before, the point-to-set mapping ψ is compact-valued and continuous on X (Lemma 4). According to Lemma 1, from $|\mathrm{Arg}\max(f, \psi(x))| = 1$ we deduce that r is upper semi-continuous point-to-point mapping. An upper semi-continuous point-to-point mapping is continuous when viewed as a function. Hence, the function r is continuous on X. The lemma is proved.

Lemma 7 [1, Proposition 2.1]**.** *If $B \subset A \subset R^m$ and B is a retract of A, then the contractibility of A implies the contractibility of B.*

This shows that contractibility of sets is preserved under retractions.

Lemma 8. $\mathrm{Max}\,(X,F)$ *is homeomorphic to* $F(\mathrm{Max}\,(X,F))$.

Proof. As it is well-known, every continuous image of a compact set is compact. In fact, the set X is compact and the function r is continuous on X. Hence, the set $\mathrm{Max}\,(X,F) = r(X)$ is compact.

Since the function $F : X \to R^n$ is continuous, it follows that the restriction $h : \mathrm{Max}(X,F) \to F(\mathrm{Max}(X,F))$ of F is continuous too. Applying Lemma 2, we deduce that if $x, y \in \mathrm{Max}(X,F)$ and $x \neq y$, then $h(x) \neq h(y)$. As a result we obtain that the function h is bijective. Consider the inverse function $h^{-1} : F(\mathrm{Max}(X,F)) \to \mathrm{Max}(X,F)$ of h. As it was proved before, the set $\mathrm{Max}\,(X,F)$ is compact, therefore, the function h^{-1} is continuous too. Finally, we obtain that the function h is homeomorphism. The lemma is proved.

We are now in a position to prove the main result of this paper.

Proof of Theorem 1. (a) From Lemmas 1, 3 and 6 it follows that there exists a continuous function $r : X \to \mathrm{Max}\,(X,F)$ such that $r\,(X) = \mathrm{Max}\,(X,F)$ and $r\,(x) = \mathrm{Arg}\max\,(f, \psi(x))$ for all $x \in X$.

(b) This follows by applying of Theorem 1(a), and Lemmas 7 and 8. The theorem is proved.

We have shown the contractibility of the Pareto solutions sets in concave vector maximization under the condition that at least one component of the function F is strictly quasi-concave. The proof of Theorem 1 cannot be extended to the general case when all the components of F are concave.

In particular, it can be seen that, by applying Lemmas 6 and 8, that both sets $\mathrm{Max}\,(X,F)$ and $F(\mathrm{Max}\,(X,F))$ are compact, path-connected and connected.

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СВИВАЕМОСТ НА МНОЖЕСТВАТА НА РЕШЕНИЯТА ПО ПАРЕТО ВЪВ ВДЛЪБНАТАТА ВЕКТОРНАТА МАКСИМИЗАЦИЯ

Здравко Д. Славов

В представената статия изучаваме решенията на Парето във вдлъбнатата векторна максимизация с компактна и изпъкнала област. Един от най-важните проблеми в тази оптимизация е изследването на топологичната структура на множествата от решенията по Парето. Ние разглеждаме проблема за конструиране на ретракция от допустимата област върху оптималното множество по Парето, ако критериалните функции са вдлъбнати и една от тях е строго квази-вдлъбната върху допустимото множество. Използвайки този резултат доказваме също, че двете множества – оптималното по Парето и фронтът на Парето са хомеоморфни и свиваеми.