CLASSIFICATION OF THE 2-SPREADS OF $PG(5, 2)^*$

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The 2-spreads of $PG(5, 2)$ are constructed and tested for equivalence by a method considering the specific properties of the automorphism group of the projective space. It is established that up to the equivalence there exists only one 2-spread of $PG(5, 2)$. The order of the automorphism group preserving it is $10584$.

1. Introduction. For the basic concepts and notations concerning combinatorial designs and projective spaces, the reader is referred, for instance, to [1], [2], [3], [5], [6], or [10].

1.1 Projective spaces and spreads. A projective space is a geometry consisting of a set of points and a set of lines, where each line is a subset of the point set, such that the following axioms hold:

- Any two points are on exactly one line.
- Let $A, B, C, D$ be four distinct points no three of which are collinear. If the lines $AB$ and $CD$ intersect each other, then the lines $AD$ and $BC$ also intersect each other.
- Any line has at least 3 points.

Let $V$ be a vector space of dimension $d + 1$ over the division ring $F$. The geometry $P(V)$ that has as its points the 1-dimensional subspaces of $V$ and as its lines the 2-dimensional subspaces of $V$, is a projective space. Any projective space that is not a projective plane is isomorphic to some $P(V)$, which is also denoted by $PG(d, F)$. If $F$ is a finite field with $q$ elements, then the notation $PG(d, q)$ is used, where $d$ is called dimension, and $q$ order of the projective space, and any line has $q + 1$ points.

An automorphism of $PG(d, q)$ is a bijective map of the point set that preserves collinearity, i.e., maps the lines into lines.

A linear subspace of a projective space is a set $U$ of points such that if $A, B \in U$, then any point on the line $AB$ is contained in $U$. Any subspace together with the lines contained in it, is a projective space. For two lines $A$ and $B$ of $PG(d, q)$, denote by $(A, B)$ the subspace of smallest dimension containing them.

A $t$-spread in $PG(d, q)$ is a set $S$ of $t$-dimensional subspaces such that any point of the geometry is on exactly one element of $S$.

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A partial t-spread in $PG(d, q)$ is a set of disjoint t-dimensional subspaces. A partial t-spread in $PG(d, q)$ is maximal if it is not properly contained in any partial t-spread of $PG(d, q)$. In this context a partial t-spread in $PG(d, q)$ which forms a partition of the points, is called a t-spread.

Two partial t-spreads in $PG(d, q)$ are equivalent if there is an automorphism of the geometry, mapping one to the other.

1.2. Combinatorial designs. Let $V = \{P_i\}_{i=1}^v$ be a finite set of points, and $B = \{B_j\}_{j=1}^b$ a finite collection of $k$-element subsets of $V$, called blocks. $D = (V, B)$ is a design with parameters $t-(v,k,\lambda)$ if any $t$-subset of $V$ is contained in exactly $\lambda$ blocks of $B$.

Two designs are isomorphic if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence.

An automorphism is an isomorphism of the design onto itself. The set of all automorphisms of a design forms a group called its full group of automorphisms. Each subgroup of this group is a group of automorphisms of the design.

A 2-(4m − 1,2m − 1,m − 1) design is called a Hadamard 2-design. A 2-(v,3,1) design is called a Steiner triple system of order $v (STS(v))$.

1.3. $PG(5,2)$-automorphism group, subspaces and related designs. There are 63 points and 651 lines in $PG(5,2)$. A $g$-dimensional subspace has $2^{g+1} - 1$ points and $(2^{2g+1} + 1)/3 - 2^g$ lines.

All the 651 lines form an $STS(63)$. The 2-dimensional subspaces of $PG(5,2)$ form a 2-(63,7,15) design, the 3-dimensional subspaces form a 2-(63,15,35) design and the 4-dimensional subspaces form a Hadamard 2-(63,31,15) design.

Any two intersecting lines of $PG(5,2)$ determine a 2-dimensional subspace.

The full automorphism group of $PG(5,2)$ is the projective general linear group $PGL(6,2)$. This group is doubly transitive both on the points and on the lines. An automorphism fixes either 0 or 1 point, or all the points of a subspace. For each subspace $P$ there is a subgroup fixing its points, and this subgroup is transitive on the points outside $P$. Thus, the order of the full automorphism group is

$|PGL(6,2)| = 63(63 − 1)(63 − 3)(63 − 7)(63 − 15)(63 − 31) = 20158709760$.

1.4. Known classifications of spreads in $PG(d, q)$. Soicher [9] classified partial 1-spreads of lines in finite projective spaces using the GRAPE package within the GAP system. His results include the construction up to equivalence of all partial 1-spreads in $PG(3,2)$, $PG(4,2)$ and $PG(3,3)$, all maximal partial 1-spreads in $PG(3,4)$ and the maximal partial 1-spreads in $PG(3,7)$ of size 45 and invariant under a group of order 5. Then, Blokhuis, Brouwer and Wilbrink classified all maximal partial 1-spreads of size 45 in $PG(3,7)$ [4].

Looking for affine 2-(64,16,5) designs of small rank, Mavron, McDonough and Tonchev constructed [8] more than 30000 1-spreads of $PG(5,2)$. Recently, Mateva and Topalova [7] classified, up to equivalence, all 1-spreads of $PG(5,2)$.

1.5. The present work. The subject of this paper are the 2-spreads of $PG(5,2)$ whose nine elements are disjoint 2-dimensional subspaces, each of which has 7 points. We use design approach to the problem. We actually make all the computations on the related to $PG(5,2)$ designs, namely, we choose the nine spread elements among the 1395 blocks of 164
the 2-(63,7,15) design, and compute the needed automorphism groups as automorphism groups of the Hadamard 2-(63,31,15) design. We use the specific properties of $PGL(6,2)$, and of the subspaces of $PG(5,2)$ to reduce the search space, to filter away the equivalent spreads, and to establish the uniqueness of the 2-spread.

2. Construction of all inequivalent spreads. Any two intersecting lines $A$ and $B$ of $PG(5,2)$ determine a 2-dimensional subspace $\langle A, B \rangle$. Denote a line through the points $a, b$ and $c$ by $\{a, b, c\}$. Without loss of generality we can denote the points of $PG(5,2)$ by the numbers $1, 2, \ldots, 63$ in such a way that $\{1, 2, 3, 4, 5, 6, 7\} = \langle \{1, 2, 3\}, \{1, 4, 5\} \rangle = P_1$ and $\{8, 16, 17, 32, 33, 34, 35\} = \langle \{8, 16, 17\}, \{8, 32, 33\} \rangle = P_2$ are 2-dimensional subspaces, $\{1, 2, \ldots, 15\}$ is a 3-dimensional subspace, and $\{1, 2, \ldots, 31\}$ is a 4-dimensional subspace. In our notations, the related Hadamard 2-(63,31,15) design is presented in Table 1. It contains all the information about the geometry, namely 3 points are on a line if they are together in 15 blocks of this design, 7 points form a 2-dimensional subspace if they are together in 3 blocks, and the 31 points of each block form a 4-dimensional subspace.

The automorphism group $G$ of the projective space is doubly transitive on the lines, and, thus, each pair of intersecting spread lines can be mapped by some automorphism into the pair $\{1, 2, 3\}$ and $\{1, 4, 5\}$. Respectively, each 2-dimensional subspace can be mapped into the space $P_1$.

For each pair of intersecting lines $X$ and $Y$, which have no point of $P_1$, there is an automorphism fixing the points of $P_1$, and mapping $X$ into $\{8, 16, 17\}$, and $Y$ into $\{8, 32, 33\}$. Therefore, for each 2-dimensional subspace $P$ containing no points of $P_1$, there is an automorphism fixing the points of $P_1$, and mapping $P$ into the 2-dimensional subspace $P_2$.

Thus, for each spread, there exists an automorphism mapping it into a spread containing the two elements $P_1$ and $P_2$. That is why we only construct spreads containing these two elements.

We realize a backtrack search. If there are already $n$ elements in the spread, we choose the $(n + 1)$-st one among the 2-dimensional subspaces containing the first point, which is in none of the $n$ spread elements. We arrange the 2-dimensional subspaces in lexicographic order defined on the numbers of the points they contain. Thus, the spread elements are lexicographically ordered, and any spread we construct is lexicographically greater than the ones constructed before it.

The main problem here is the very big number of isomorphic solutions. Obtaining a spread, we have to check for the existence of an automorphism mapping it into an already constructed one, i.e. into a lexicographically smaller one. However, $PG(5,2)$ has 20158709760 automorphisms, and putting in order all of them makes the computation time impossible.

The number of automorphisms which can map the constructed spreads into one another, is actually smaller, because the first two elements in them are the same.

Let $S_1$ and $S_2$ be two equivalent spreads, and let $\alpha \in G$ map the elements of $S_2$ into the elements of $S_1$. Let $\alpha Q = P_1$ and $\alpha R = P_2$, where $Q, R \in S_2$. Suppose there exists $\beta \in G$, such that $\beta Q = P_1$ and $\beta R = P_2$.

There exists $\varphi \in G$, such that $\alpha = \varphi \beta$. Then, $\varphi = \alpha \beta^{-1}$. Therefore, $\varphi$ fixes $P_1$ and $P_2$. 

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Table 1. The 2-(63,31,15) design of the hyperplanes of $PG(5, 2)$

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10000111110111011001000010110101000100010100111101001010111011101101110110101011101
01001101110011010111011101110110101000101101110001011010101100110101011010101101
0111001011011001011011100100010111001110010101110100101001100101001111010101111101
111001111100101101011011001011101000000101101001110011011011100000111010011010110
11011011110011010101110100101101001100110111001001001101011010011101111111011000
10110111111100110010001101101010100110101001010110100101011010010100101111001010
1001011010100110111001010001001101110110100100110101001010011011111011011011101
1110111111111111111111111111111111111111111111111111111111111111111111111111111111
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Before starting the spread construction, for each 2-dimensional subspace of $PG(5, 2)$ we find and save an automorphism that maps it into $P_1$. For each disjoint with $P_1$ 2-dimensional subspace we find and save an automorphism that fixes $P_1$ and maps it into $P_2$.

When we obtain a new spread, we check if one of the automorphisms $\varphi, \delta, \gamma$ maps it into a lexicographically smaller one. Here $\gamma_i$ is an automorphism mapping the spread element $P_i$ into $P_1$, $\delta_j$ is an automorphism fixing $P_1$ and mapping the 2-dimensional subspace $\gamma_i Q_j$ into $P_2$, and $\varphi_n$ is an automorphism fixing the 2-dimensional subspaces $P_1$ and $P_2$. Here $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 8$. The automorphism group preserving $P_1$ and $P_2$ is of order 28224, and, thus, $n = 1, 2, \ldots, 28224$.

This way instead of trying all the 20158709760 automorphisms, to check for equivalence we use 2032128 = 9.8.28224 of them, and the computation becomes possible.

3. Results. We obtain 192 2-spreads, containing $P_1$ and $P_2$ as two of their nine elements. For each two of them, there is an automorphism among those 2032128 described above, which maps them into one another. So they are all equivalent to the following 2-spread:

{1, 2, 3, 4, 5, 6, 7}, \{8, 16, 17, 32, 33, 34, 35\}, \{9, 18, 24, 36, 37, 45, 62\},
{10, 19, 27, 44, 50, 52, 63}, \{11, 25, 31, 38, 48, 51, 56\}, \{12, 23, 30, 39, 43, 47, 55\},
{13, 20, 22, 42, 46, 54, 61}, \{14, 26, 28, 41, 49, 57, 58\} \{15, 21, 29, 40, 53, 59, 60\}.

We next find out the subgroup of $PGL(6, 2)$ which maps the spread into itself. It is of order 10584.

4. Correctness of the results. The construction and classification of the spreads, and the computation of the automorphism groups was done by our own C++ programs, some of them written, and others modified for this specific task. Since programming mistakes are always possible, we obtained by two different programs the number (192) of all 2-spreads, containing the elements $P_1$ and $P_2$, and the order (28224) of the automorphism group fixing $P_1$ and $P_2$.

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КЛАСИФИКАЦИЯ НА 2-СПРЕДОВЕТЕ НА $PG(5, 2)$

С помощта на компютър построяваме 2-спредовете на $PG(5, 2)$ и ги тестваме за еквивалентност по метод, използващ специфичните свойства на групата от автоморфизми на проективната геометрия. Получаваме, че с точност до еквива lentност съществува един единствен 2-спред на $PG(5, 2)$. Групата от автоморфизми, която го запазва, е от ред 10584.