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GENERALIZED FRACTIONAL CALCULUS, SPECIAL FUNCTIONS AND INTEGRAL TRANSFORMS: WHAT IS THE RELATION?*

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In this survey we briefly illustrate some of our contributions to the generalizations of the fractional calculus (analysis) as a theory of the operators for integration and differentiation of arbitrary (fractional) order, of the classical special functions and of the integral transforms of Laplace type. It is shown that these three topics of analysis are closely related and mutually induce their origins and developments. Due to the short space, we confine here only to survey the ideas of our recent contributions related to the title. Statements of the numerous results, their proofs, examples and applications can be found in Refs, like: [1]–[2], [5]–[7], [11]–[19].

1. Preliminaries. The notion "Fractional Calculus" (FC) or "Fractional Analysis" is used for the extension of the Calculus (Analysis), when the order of integration and differentiation can be an arbitrary number (fractional, irrational, complex), that is, not obligatory integer. The definitions of the operators for fractional integration and differentiation, by extension either of the formula for repeated integration, or of the finite differences approach, provide pedagogical examples for the ways of generalization in Maths, so to achieve a further reasonable wider notion and higher extent of freedom. Namely, it was enough to replace the "n-factorial" in the Cauchy formula

$$R^n f(z) := D^{-n} f(z) = \int_0^z dt_1 \int_0^{t_1} \dots \int_0^{t_{n-1}} f(t_n)\, dt_n = \frac{1}{(n-1)!} \int_0^z (z-t)^{n-1} f(t) dt, \quad n \in \mathbb{N}_+,$$

by a "Gamma-function", so to achieve the so-called *Riemann-Liouville (R-L) integral* (operator for integration) of fractional order $\delta > 0$ (or $\Re(\delta) > 0$)

$$(1) \qquad R^\delta f(z) := D^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1} f(t) dt = z^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(z\sigma) d\sigma\,.$$

Then, on the basis of the axiom $DRf(z) = f(z)$, $D^n R^n f(z) = f(z)$ (where R and $D = \dfrac{d}{dz}$ denote the integration and differentiation of 1st order), it is reasonable to

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define the corresponding *Riemann-Liouville fractional derivative* as a composition of a derivative of integer order and an integral of positive fractional order of the form (1):

$$D^\delta f(z) := D^n R^{n-\delta} f(z) = \left(\frac{d}{dz}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^z (z-t)^{n-\delta-1} f(t)dt \right\}, \tag{2}$$

where $n := [\delta] + 1 > \delta$, $[\delta] =$ the integer part of δ. A little more general, as depending on 3 parameters $\delta > 0$ (order), $\gamma \in \mathbb{R}$ (weight) and $\beta > 0$, are the *Erdélyi-Kober (E-K) fractional integration operators*

$$I_\beta^{\gamma,\delta} f(z) = \left[t^{-(\gamma+\delta)} R^\delta t^\gamma f(t^{\frac{1}{\beta}})\right]_{t:=z^\beta} = \frac{z^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^z (z^\beta - \xi^\beta)^{\delta-1} \xi^{\beta\gamma} f(\xi)\, d(\xi^\beta),$$

which for treatment, applications and generalizations are more suitable to be written, after a substitution, as:

$$I_\beta^{\gamma,\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-\sigma)^{\delta-1} \sigma^\gamma f(z\sigma^{\frac{1}{\beta}})\, d\sigma\,. \tag{3}$$

The corresponding *E-K fractional derivatives* have a symbolical representation as

$$D_\beta^{\gamma,\delta} z(z) = \left[t^{-\gamma} D^\delta t^{\gamma+\delta} x(t^{\frac{1}{\beta}})\right]_{t:=z^\beta}.$$

Development of classical FC. The theory of the FC and its first applications, in the classical variant based on (1), (2), (3) or other definitions have been described enough fully in the encyclopedia–book by Samko, Kilbas and Marichev [23]. The development of FC had been rather controversial and did not go smoothly in its "Old history" period (1695–1970), see the references available in [23] and in our notes [25], [26]. A substantial revival and extensive development has been inspired after 1970, in view of the needs for fractional order mathematical models in pure Maths, Physics and Chemistry as well as in the Applied Sciences that describe better the really existing phenomena and events from the material and social world, having a fractal nature. The open questions as "If there exist any geometrical or physical meanings of the operators of FC or at least proper mathematical models using them..." had stepped down from the agenda, replaced by the new challenges how to implement wider and more efficiently the FC tools in solving the problems, by means of fractional order differential and integral equations and systems, combined with numerical and graphical interpretations, and further development of the theory. Evidences for the extensive introduction of FC, during its "Recent history" (1970–nowadays), in various fields of science and practice, are given by the fact of introducing many new positions in the 2010 version of "Mathematics Subject Classification": 26A33, 33E12, 34A08, 34K37, 35R11, 60G22, etc., and by the long lists of conferences, books, surveys, journals' special issues, patents, algorithms and software packages etc. dedicated entirely to FC and its applications, provided in our notes [24], [26]. The interested readers can find the color posters from papers [24], [25] illustrating the old and recent history of FC, as well as the huge collection of stuff published in Vols. $1 \div 13$ (1998–2010) in the journal FCAA [9], at its websites.

Special functions. Relationships between them and FC have been hidden or emphasized yet it the applications of classical FC to some classical special functions (SF), as the Bessel, Gauss and Tricomi functions and some orthogonal polynomials. But recently, the marriage between these two parties contributed to essentially new trends of development and applications for each of them. The SF served as kernels of the operators of the generalized FC. From other side, the tools of FC were used to introduce new classes of SF as well as to develop further the theory of classical SF (SF of Mathematical Physics) and their classification, see the sketch in Sect. 3. The SF of FC (a recent notion) appear as the solutions of fractional order differential and integral equations. We need to remind first the definitions of some basic special functions, essentially lying in the generalizations of the FC.

The *Fox H-function* has been introduced in 1960 as a generalized hypegeometric function defined by means of the Mellin-Barnes type contour integral:

$$(4)\qquad H^{m,n}_{p,q}\left[\sigma\,\middle|\,\begin{matrix}(a_k,A_k)_1^p\\(b_k,B_k)_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod\limits_{k=1}^{m}\Gamma(b_k-B_ks)\prod\limits_{j=1}^{n}\Gamma\left(1-a_j+sA_j\right)}{\prod\limits_{k=m+1}^{q}\Gamma(1-b_k+sB_k)\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-sA_j\right)}\sigma^s\,ds.$$

Here $\mathcal{L}$ is a suitable contour in $\mathbb{C}$, the integer orders $0\le m\le q, 0\le n\le p$ and the parameters $a_j\in\mathbb{R}, A_j>0, j=1,\dots,p,\ b_k\in\mathbb{R}, B_k>0, k=1,\dots,q$ are such that $A_j(b_k+l)\neq B_k(a_j-l'-1),\ l,l'=0,1,2,\dots$. For the basic 3 types of contours and conditions for existence and analyticity of the H-function in disks $\subset\mathbb{C}$ whose radii are $\rho=\prod_{j=1}^{p}A_j^{-A_j}\prod_{k=1}^{q}B_k^{B_k}>0$, one can see in the recently appeared books on special functions, as [21], [13, App.E] (see other books listed in [26]). When $A_1=\dots=A_p=1, B_1=\dots=B_q=1$, (4) turns into more popular and simple *Meijer's G-function* $G^{m,n}_{p,q}\left[\sigma\,\middle|\,\begin{matrix}(a_k)_1^p\\(b_k)_1^q\end{matrix}\right]$ from the years 1940–1946, available in the Bateman-Erdélyi Project, see [8]. The G- and H-functions encompass almost all the elementary and special functions. The generalized hypergeometric functions ${}_pF_q$, and all their special cases (as the classical orthogonal polynomials, the Bessel and cyllindrical functions, Gauss and Tricomi hypergeometric functions, and other named functions of Mathematical Physics) are G-functions, see lists in [8], [21], [13, App.C]. But we have the essential examples of the *Wright* (*called also Fox-Wright*) *generalized hypergeometric function*:

$$(5)\qquad \begin{aligned}{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\,\middle|\,\sigma\right]&=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\frac{\sigma^k}{k!}\\&=H^{1,p}_{p,q+1}\left[-\sigma\,\middle|\,\begin{matrix}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{matrix}\right]\end{aligned}$$

with irrational $A_j,B_k>0$; and of the *Mittag-Leffler* (*M-L*) *function* with irrational indices $\alpha>0$:

$$(6)\qquad E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ \beta>0,$$

that are H-functions, *not* reducible to G-functions. Only for $A_1=\dots=A_p=B_1=\dots=B_q=1$, (5) reduces to const.${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;\sigma)$ which is a $G^{1,p}_{p,q+1}$-function; and (6) which generally satisfies a differential equation of fractional order α, reduces for $\alpha=1,2$ to the exponential or trigonometric function.

Integral transforms. In this survey, we briefly discuss also the many-fold connection between the FC and the integral transforms (IT). According to the recent points of view, there are three basic kinds of the so-called *convolutional IT*: i) of the type of the Fourier transform (the sin-Fourier, cos-Fourier, Hankel, symmetrical Fourier transforms); ii) of the type of the fractional integrals (as (1), Erdélyi-Kober integrals and their generalizations considered in Sect. 2); iii) of the type of the *Laplace transform*

$$\mathcal{L}\{f(z);s\} = \int_0^\infty \exp(-zs)f(z)dz, \tag{7}$$

and its generalizations (as the Meijer, Borel-Dzrbashjan and Obreshkoff transforms).

All these IT have as kernels some elementary or special functions which are, in general, Fox H-functions. The contemporary notion to be used for them is *H-transforms* (or *G-transforms*, when the kernels can be represented as the simpler Meijer G-function). In this survey we shortly discuss our contributions to the class iii) of the *generalized integral transforms of Laplace type*, that is to the H-transforms (resp. G-transforms) of the form

$$\mathcal{H}\{f(z);s\} = \int_0^\infty H_{p,q}^{m,n}\left[zs \,\middle|\, \begin{matrix}(a_j, A_j)_1^p\\(b_k, B_k)_1^q\end{matrix}\right] f(z)dz, \tag{8}$$

when

$$\Omega=\sum_{j=1}^{n} A_j-\sum_{j=n+1}^{p} A_j+\sum_{k=1}^{m} B_k-\sum_{k=m+1}^{q} B_k>0 \text{ (or } \delta = m+n-\frac{p+q}{2}>0, \text{ for } G\text{-transf.).} \tag{9}$$

From one side, the operators of the generalized FC (GFC) considered in Sect. 2 are IT of form (8) and class ii); from the other side we study IT of form (8) which are "generated" by the GFC operators on the basis of the Laplace transform. From third side, the new transforms (8) introduced in Sect. 4 algebrize some related GFC operators, have the same convolutions as them, and thus can serve for operational calculi development.

2. Generalized fractional calculus (GFC). From the years 1967-1970, some authors as Kalla, Saxena, Saigo, Love, Lowndes, etc. introduced generalizations of the classical FC operators. These are the so-called *generalized operators of fractional integration*, of the general form proposed by Kalla [10]:

$$\mathcal{I}f(z) = z^{-\gamma-1}\int_0^z \Phi\left(\frac{\xi}{z}\right)\xi^\gamma f(\xi)d\xi = \int_0^1 \Phi(\sigma)\,\sigma^\gamma f(z\sigma)d\sigma, \tag{10}$$

where the elementary functions in the kernels of the R-L and E-K integrals (1), (3) have been replaced by some special functions $\Phi(\sigma)$ – like the Bessel, Gauss, Horn, G- and H-functions. The authors aimed to choose the kernels and the functional classes for (10) in a way so to have some analogues of the *axioms of the classical FC* (the semi-group property, the zero order giving identity operator, while an integer order giving the n-fold integrals or n-th order derivatives). Not much more than few formal properties and propositions have been given in their works, with lack of whole theory and any applications. The failure should be sought in the rather particular, or in contrary – rather general choice of the special functions $\Phi(\sigma)$, not allowing to reveal their structure.

In [11], [12], [13] (see a survey also in [16]), thanks to the suitable choice (order) of the Meijer $G^{m,0}_{m,m}$- and Fox $H^{m,0}_{m,m}$-functions, we defined generalized operators of fractional integration and differentiation and developed their detailed theory (GFC) with various applications in theory of the SF, IT, in solving differential and integral equations of different kinds – including higher integer order or fractional multi-order, and other unexpected topics of Analysis and Mathematical Modeling. See the book [13] and other subsequent works. In the most general case with an H-function, we have the following

Definition 1. Let $m \geq 1$ be an integer and let $\delta_k \geq 0$, $\gamma_k \in \mathbb{R}$, $\beta_k > 0$, $k = 1,\dots,m$ be m-tuples of parameters. Consider $\boldsymbol{\delta} = (\delta_1,\dots,\delta_m)$ as *multi-order of fractional integration*, $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_m)$ as a multi-weight, and resp. $\boldsymbol{\beta} = (\beta_1,\dots,\beta_m)$ as an additional multi-index. The integral operators of the form :

(11)
$$I^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\beta},m} f(z) := I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) := \begin{cases} \int\limits_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{matrix} \right.\right] f(z\sigma)d\sigma, & \text{if } \sum\limits_{k=1}^m \delta_k > 0, \\ f(z), \quad \text{if} \quad \delta_1 = \delta_2 = \dots = \delta_m = 0, \end{cases}$$

are said to be *multiple (m-tuple, generalized) Erdélyi-Kober (E-K) operators of fractional integration*. More generally, each operator of the form

$$I f(z) = z^{\delta_0}\, I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z), \tag{12}$$

with arbitrary $\delta_0 \geq 0$, is called shortly *generalized (m-tuple) fractional integral*.

The generalized fractional integrals of multi-order $\boldsymbol{\delta} = (\delta_1 \geq 0,\dots,\delta_m \geq 0)$, basic for our GFC, are defined by means of single integrals, but they happen to be also *compositions of $m>1$ commuting Erdélyi-Kober fractional integrals* (3) ($k = 1,\dots,m$)

(13)
$$I^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\beta},m} f(z) = \left[\prod_{k=1}^m I^{\gamma_k,\delta_k}_{\beta_k}\right] f(z) = \underbrace{\int_0^1 \dots \int_0^1}_{m} \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k-1}\sigma_k^{\gamma_k}}{\Gamma(\delta_k)}\right] f(z\sigma_1^{\frac{1}{\beta_1}}\dots\sigma_m^{\frac{1}{\beta_m}})\, d\sigma_1\dots d\sigma_m,$$

written without any use of special functions. Definition (11) uses the priority of condense and efficient symbolical manipulations and the powerful tools of the H- (or G-functions), while *decomposition* (13) gives the *key to numerous applications of our GFC operators.*

Examples. The case $m = 1$ gives the E-K operators (3), and in particular the Riemann-Liouville fractional integrals (1), as well as all their special cases widely used in analysis. For $m = 2$, the $H^{2,0}_{2,2}$-function reduces to a hypergeometric function ${}_2F_1$ and from (11) we get the *hypergeometric fractional integrals* involving the Gauss function. Some examples of our GFC operators for $m = 3$, including Horn's F_3-functions have been studied by Marichev, Saigo, etc. For arbitrary $m > 1$ we have from the operators of GFC, the *hyper-Bessel operators of Dimovski* (see [3], [13, Ch.3]): $B = D^{(\gamma_k),(1,\dots,1)}_{(\beta,\dots,\beta),m} z^{-\beta}$, $L = z^{\beta} I^{(\gamma_k),(1,\dots,1)}_{(\beta,\dots,\beta)m}$. Many other generalized differentiation and integration operators are special cases in the scheme of GFC, some of them – in univalent function theory, others – as transmutation operators for solving differential and integral equations, in operational calculus, or in representation formulas for special functions.

We omit the details for the corresponding functional spaces (of weighted continuous, integrable or analytic functions) and the appropriate conditions on parameters on which the operators of GFC are bounded linear operators mapping a function class int itself.

Operational rules. Just a short list of them, giving idea for the axioms of GFC and for the efficient usage of the tools of the H- and G-functions as kernels:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{z^p\} = c_p z^p, \text{ with } c_p = \prod_{k=1}^m [\Gamma(\gamma_k+1+p/\beta_k)/\Gamma(\gamma_k+\delta_k+1+p/\beta_k)];$$

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} z^\lambda f(z) = z^\lambda I^{(\gamma_k+\lambda/\beta),(\delta_k)}_{(\beta_k),m} f(z) \quad \text{(commutability with power functions);}$$

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\beta_k),m} f(z) = I^{(\tau_j),(\alpha_j)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = I^{(\gamma_k,\tau_j),(\delta_k,\alpha_j)}_{(\beta_k),m+n} f(z)$$

(commutability and compositions of operators (11));

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m} f(z) \quad \text{(product rule, semigroup property);}$$

$$\left\{ I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \right\}^{-1} f(z) = I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m} f(z) := D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z); \quad \text{(inversion formula).}$$

Generalized fractional derivatives and differ-integrals. On the basis of the latter formal inversion, by means of fractional "integrals" of negative multi-order, the corresponding *generalized fractional derivatives* $D^{\boldsymbol{\delta},\boldsymbol{\gamma}}_{\boldsymbol{\beta},m} := D^{(\gamma_k),(\delta_k)}_{\beta,m}$ are defined by explicit integro-differential representations, coming from the idea of the classical R-L derivatives (2), but involving polynomials of the differentiation $z\dfrac{d}{dz}$ and integrals (11) of positive multi-orders (see definitions and properties in [11], [12], [13], [16]). Next, we are able to consider the generalized fractional integrals and derivatives as one single object, called *generalized fractional differ-integrals*, by assigning to each symbol $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ with arbitrary (real) orders δ_k a suitable composition of classical E-K derivatives, integrals and identity operators: if $\delta_1 < 0, \dots, \delta_s < 0$; $\delta_{s+1} = \dots = \delta_r = 0$; $\delta_{r+1} > 0, \dots, \delta_m > 0$, we can assume

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} = \left[\prod_{k=1}^{s} D^{\gamma_k+\delta_k,-\delta_k}_{\beta_k} \right] \cdot \left[\underbrace{I \dots I}_{(r-s)} \right] \cdot \left[\prod_{j=r+1}^{m} I^{\gamma_j,\delta_j}_{\beta_j} \right] = D^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),s} \, I^{(\gamma_j),(\delta_j)}_{(\beta_j),m-r} .$$

3. Applications to special functions. Usually, the special functions of mathematical physics, being particular cases of the ${}_pF_q$*-functions* (and thus also of (5)):

$$(14) \qquad {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (a)_k := \frac{\Gamma(a+k)}{\Gamma(a)},$$

provided $|z| < \infty$ if $p \leq q$, and $|z| < 1$ if $p = q+1$, are defined by means of their power series representations, like (14). However, some alternative representations can be used as their definitions, as: the well-known *Poisson integrals* for the Bessel functions and the analytical continuation of the Gauss hypergeometric function (g.h.f.) *via* the *Euler integral formula*. Also, some special functions have representations by *repeated*, or even *fractional order differentiation*. Such examples are given by the *Rodrigues differential formulas* for the classical orthogonal polynomials (see Rusev [22]), used often as their basic definitions. There exists various such integral and differential formulas, quite peculiar for each corresponding special function and scattered in the literature without any common idea.

In [13], [14] we have proposed an *unified approach in deriving all such formulas and their generalizations* (new or newly written integral, differential and differ-integral representations of the special functions) by means of the generalized fractional calculus (GFC). A suitable classification of these special functions is also introduced. *The idea is based on the following simple facts*: i) most of the special functions of mathematical physics are nothing but modifications of the g.h. f-s ${}_pF_q$; ii) each ${}_pF_q$-function can be represented as an E.-K. fractional differintegral of a ${}_{p-1}F_{q-1}$-function; iii) a finite number (q) of steps ii) leads to one of the basic g.h.f-s: ${}_0F_{q-p}$ (for $q-p=1$: Bessel function), ${}_1F_1$ (confluent h.f.), ${}_2F_1$ (Gauss h.f.); iv) these 3 basic g.h.f-s can be considered as fractional differintegrals of 3 elementary functions, depending on if $p<q$, $p=q$ or $p=q+1$, viz.:

$$\cos_{q-p+1}(z), \quad z^{\alpha} \exp z, \quad z^{\alpha}(1-z)^{\beta}; \tag{15}$$

v) the compositions of E.-K. operators arising in iii) give generalized (q-tuple) fractional integrals (11), or corresponding generalized fractional derivatives. Thus we obtain the following basic proposition:

Proposition 1. *All the generalized hypergeometric functions ${}_pF_q$, and, therefore, all the classical special functions, can be considered as generalized (q-tuple) fractional differintegrals $I^{(\gamma_k),(\delta_k)}_{(\beta,\ldots,\beta),q}$, $D^{(\gamma_k),(\delta_k)}_{(\beta,\ldots,\beta),q}$ of one of the elementary functions* (15)*, depending on if $p<q$, $p=q$, $p=q+1$.*

In the denotation in (15), $y(z)=\cos_m(z)$ stands for the so-called *generalized cosine function*, i.e. the solution of the Cauchy problem

$$y^{(m)}(z)=-y(z), \quad y(0)=1, y^{(k)}(0)=0,\ k=1,\ldots,m-1, \quad \text{namely:}$$

$$\cos_m(z) := {}_0F_{m-1}\left(\left(\frac{k}{m}\right)_1^{m-1}; -\left(\frac{z}{m}\right)^m\right) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{mk}}{(mk)!}\,. \tag{16}$$

Practically, behind the general saying in Proposition 1 there are hidden several lemmas and theorems, see their statements and details in [13], [14]. They provide new integral and differential formulas for the special functions, some of them generalizing the Poisson or Euler integral formulas, others – the Rodrigues differential formulas; suggest also some numerical algorithms for their calculation. Each of them proposes a new sight on the generalized hypergeometric functions (gen.h.f-s) ${}_pF_q$ of the *three classes* formed in such a way, and called respectively: 1) $p<q$: *gen.h.f-s of Bessel type* (the $\cos_m$-function, the Bessel function, the hyper-Bessel function (13), etc.); 2) $p=q$: *gen.h.f-s of confluent type* (e.g. the exp-function, Kummer confluent function ${}_1F_1$, Laguerre polynomials, error function, etc.); 3) $p=q+1$: *gen.h.f-s of Gauss type* (the Gauss function ${}_2F_1$, the Jacobi, Legendre etc. polynomials). Such a scheme proposes, especially to applied scientists and engineers, a more comprehensive view on the complicated special functions - like GFC-images of the simplest functions (15), with similar to their properties (at least in asymptotical or pedagogical sense). Studying of the special functions can be simplified, by their *"reduction" via the GFC operators to few simpler functions*: elementary ones like (15), or much better *known special functions*, like the Bessel, Kummer and Gauss functions. From these comments, we can observe the special role that the H- and G-functions play, even *among the other special functions from which they have emerged as generalizations* and the close relationships between the topics of FC and SF.

Further, to Proposition 1, exploring operators of GFC with all equal parameters $\beta_1=\beta_2=\cdots=\beta_m:=\beta$, involving G-functions and concerning only the ${}_pF_q$-functions (that is, the classical SF), we propose *similar proposition for the so-called special functions of fractional calculus* (SF of FC). It states that all the Wright generalized hypergeometric functions (5) can be represented as GFC operators of one of the simpler functions

$$\cos_{q-p+1}(z)\ (\text{for } p<q)\,,\ \ z^{\alpha}\exp z\ (\text{for } p=q)\,,\ \ {}_1\Psi_0\,(z;(\beta,B))\ \ (\text{for } p=q+1). \tag{17}$$

The detailed statements and their proofs are given in the recent papers [17], [18].

Another relationship between the FC and SF is revealed by the new class of SF, appearing as *multi-index analogues of the classical Mittag-Leffler function* (6), the so-called Queen-function of the FC. In our papers [15], [17] we have introduced and studied the following SF, which are also fractional-indices analogues of the hyper-Bessel functions, related to the hyper-Bessel operators of Dimovski ([3], see in [13, Ch.3], [7]).

Definition 2. Let $m > 1$ be integer, $\rho_1, \dots, \rho_m > 0$ and $\mu_1, \dots, \mu_m$ be arbitrary real parameters. By these "multi-indices" $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$, we define the *multi-index Mittag-Leffler function (multi-M-L)*:

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z) := E^{(m)}_{\mathbf{1}/\boldsymbol{\rho},\boldsymbol{\mu}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1) \dots \Gamma(\mu_m + k/\rho_m)} \,. \tag{18}$$

Proposition 2. *The multi-index M-L function* (18) *is an entire function of order* ρ *and type* σ*, given as follows:*

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \dots + \frac{1}{\rho_m}, \quad \sigma = \left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}} \dots \left(\frac{\rho_m}{\rho}\right)^{\frac{\rho}{\rho_m}} > 1 \quad (\textit{for } m > 1).$$

Setting $\mu = \mu_1 + \dots + \mu_m$*, we obtain the following asymptotic estimate:*

$$|E_{(\frac{1}{\rho_i}),(\mu_i)}(z)| \le C|z|^{\rho((1/2)+\mu-(m/2))} \exp\left(\sigma |z|^{\rho}\right), \quad |z| \to \infty.$$

These are typical representatives of the SF of FC, which are Wright functions and H-functions as well ([15]):

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = {}_1\Psi_m\left[\begin{array}{c} (1,1) \\ (\mu_i, \frac{1}{\rho_i})_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1}\left[-z \middle| \begin{array}{c} (0,1) \\ (0,1), (1-\mu_i, \frac{1}{\rho_i})_1^m \end{array}\right],$$

and satisfy *important class of differential equations* (see all details in [15], [2], [17])

$$\mathcal{D} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \lambda E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z), \quad \text{of fractional multi-order } \frac{\mathbf{1}}{\boldsymbol{\rho}} = \left(\frac{1}{\rho_1}, \dots, \frac{1}{\rho_m}\right),$$

where $\mathcal{D} = z^{-1} D^{\boldsymbol{\mu}-\mathbf{1}-\mathbf{1}/\boldsymbol{\beta},\, \mathbf{1}/\boldsymbol{\beta}}_{\boldsymbol{\beta},m}$ is generalized fractional derivative.

Examples. Obviously, for $m=1$ (18) reduces to the classical M-L function (6), some examples of which are: the Rabotnov function $z^{\alpha} E_{\alpha,\alpha}(z^{\alpha})$, the Lorenzo-Hartley function, the error and the incomplete gamma functions, exponential and trigonometric functions.

For $m = 2$ the function (18), denoted as $\Phi_{\rho_1,\rho_2}(z; \mu_1, \mu_2)$, was studied by Dzrbashjan in 1960. Its examples are the Bessel function $J_{\nu}(z)$, the Struve and Lommel functions $s_{\mu,\nu}(z), s_{\nu,\nu}(z)$, the Bessel-Maitland function $J^{mu}_{\nu}(z)$, known also as the Wright function $\phi(\mu, \nu + 1; z)$, the Mainardi function, the Airy function, the Krätzel function, as well as the recently studied functions named as Wright-Bessel-Lommel functions $J^{\mu}_{\nu,\lambda}(z)$, $J^{r,n}_{\nu,\lambda}(z)$.

For arbitrary $m > 1$, we get e.g. the hyper-Bessel function of Delerue and its fractional-indices analogue, the multi-index Rabotnov function, etc.

All these SF have appeared in studying processes of diffusion, control theory or in media with fractal nature, modeled by the fractional diffusion-wave differential equations, and other fractional order equations and systems arising from science and practice.

4. Generalizations of the Laplace transform. In this section we briefly review some applications of the GFC to generalized integral transforms (8) of Laplace type.

The idea is to obtain rather general convolutional integral transforms and to study their basic properties by means of the *transmutation method.* This approach arises in

Mathematics from the natural striving to find solution of a new problem by means of reduction it to a simpler problem whose solution is already known or easier to find; to reduce the study of some new object to the known facts for another simpler object, etc., by means of the so-called transmutation operator, i.e. a suitable transformation that plays a role of translator. We have used this approach widely, for obtaining solutions of fractional or higher order differential equations (as the hyper-Bessel equations) from the solutions of same order equations but of the simplest form, see e.g. [13], [2], etc.

With respect to the integral transforms (IT), we consider the *Laplace transform* (7) *as a model IT*, from which by means of transmutation operators that are taken as operators of GFC, we generate new IT. The composition of the Laplace transform by a generalized fractional integral T of the form (12),

$$(19) \quad \mathcal{LT}\{f(z);s\} := L\{Tf(z);s\} = \int_0^\infty \exp(-zs)\, Tf(z)dz, \quad \text{with } T = z^{\lambda_0} I^{(\gamma_k),(\lambda_k)}_{(\beta_k),m} f(z),$$

we call a *generalized Laplace type integral transform, generated by the transmutation operator T*. It is proved (T.5.6.4, [13, Ch.5]) that (19) is a an H-transform (8), for which the operation $\left(f \overset{T}{*} g\right)(z) = T^{-1}\{(Tf * Tg)(z)\}$ is a convolution, where $(*)$ denotes the (Duhamel) convolution for the Laplace transform. For these notions, see Dimovski [4].

One of the integral transforms arising as interpretation of (19), and seeming to be the most general transform of Laplace type studied by now, is the following one, introduced and studied in [15], [1], etc.

Definition 3. Let $m \geq 1$ be integer, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, with $\rho_i > 0, \mu_i \in \mathbb{R}, i = 1, \ldots, m$. Then the H-integral transformation of Laplace type

$$(20) \quad \mathcal{B}^{(m)}_{\boldsymbol{\rho},\boldsymbol{\mu}}(s) = \mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\} = \int_0^\infty H^{m,0}_{m,m}\left[zs \,\middle|\, \begin{matrix} - \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_{i=1}^m \end{matrix}\right] f(z)dz, \; f \in \mathcal{H}_{\boldsymbol{\mu},\boldsymbol{\rho}},$$

is called a *multi-index Borel-Dzrbashjan transform.* For the class $\mathcal{H}_{\boldsymbol{\mu},\boldsymbol{\rho}}$, see in [15], [1].

It is closely related to the multi-index M-L function (18), and to the operators of GFC. Namely, there is a notion of the so-called *Gelfond-Leontiev (G-L) operators of generalized differentiation and integration with respect to an entire function* $\varphi(\lambda) = \sum_0^\infty \varphi_k \lambda^k$ such that $\varphi_k \neq 0$, $k = 0, 1, 2, \ldots$. For an analytic function $f(z) = \sum_0^\infty a_k z^k$ these are defined as follows: $D_\varphi f(z) = \sum_{k=1}^\infty a_k \dfrac{\varphi_{k-1}}{\varphi_k} z^{k-1}$, $I_\varphi f(z) = \sum_{k=0}^\infty a_k \dfrac{\varphi_{k+1}}{\varphi_k} z^{k+1}$, by means of the coefficients φ_k in the respective circular domains $|z| < R$ (see details e.g. in [13, Ch.2]). Considering the G-L operators with respect to the function $E_{(\frac{1}{\rho_i}),(\mu_i)}(z)$ with coefficients φ_k as in (18), we denote them by $D_{\boldsymbol{\rho},\boldsymbol{\mu}}, L_{\boldsymbol{\rho},\boldsymbol{\mu}}$ and prove that these are extendable to starlike domains containing the origin as GFC operators: $L_{\boldsymbol{\rho},\boldsymbol{\mu}} = z\, I^{(\mu_i-1),(1/\rho_i)}_{(\rho_i),m}$, resp. $D_{\boldsymbol{\rho},\boldsymbol{\mu}} = z^{-1}\, I^{(\mu_i-1-1/\rho_i),(1/\rho_i)}_{(\rho_i),m}$.

Proposition 3. *Let $\rho_i > 0, \mu_i > 0$, $i = 1, \ldots, m$ and $f(z)$ be an analytic function of the class $\mathcal{H}_{\boldsymbol{\mu},\boldsymbol{\rho}}$. Then, the multi-index Borel-Dzrbashjan transform has the property to algebrize the action of the G-L generalized differentiation and integration, as follows:*

$$\mathcal{B}^{(m)}_{\boldsymbol{\rho},\boldsymbol{\mu}}\{L_{\boldsymbol{\rho},\boldsymbol{\mu}}f(z);s\}=\frac{1}{s}\,\mathcal{B}^{(m)}_{\boldsymbol{\rho},\boldsymbol{\mu}}\{f(z);z\},\ \ \mathcal{B}^{(m)}_{\boldsymbol{\rho},\boldsymbol{\mu}}\{D_{\boldsymbol{\rho},\boldsymbol{\mu}}f(z);s\}=s\,\mathcal{B}^{(m)}_{\boldsymbol{\rho},\boldsymbol{\mu}}\{f(z);z\}-f(0).\mathrm{const},$$

and the transformation has the same convolution as the generalized integration $L_{\boldsymbol{\rho},\boldsymbol{\mu}}$.

Clearly, it is seen the analogy with the role of the model Laplace transform in the operational calculus for the classical differentiation and integration. For this sake, the images of some basic elementary and special functions are found, and several complex and real inversion formulas for (20) are found ([15], [1]).

Examples. Let $m = 1$, and the transmutation operator T in (19) be the simple transformation $Tf(z) = z^{\mu-1}f(z^{1/\rho})$. Then, the corresponding Laplace type transform is a modification of the so-called *Borel-Dzrbashjan transform* (studied in [6], [13, Ch.2]):

$$\mathcal{B}_{\rho,\mu}\{f(z);s\} = s^{\mu\rho-1}L\{Tf(z);s^{\rho}\} = \rho s^{\mu\rho-1}\int_0^{\infty}\exp(-z^{\rho}s^{\rho})\,z^{\mu\rho-1}f(z)dz.$$

For $m = 1$, $\mu = \rho = 1$, or for arbitrary $m > 1$ but all $\lambda_0 = \lambda_1 = \cdots = \lambda_m = 0$ in (19), the transmutation reduces to the identity: $T = Id$, and we have the simplest *Laplace transform.* When $m = \rho_1 = \rho_2 = 2, \mu_{1,2} = \mp\frac{\nu}{2} + 1, \lambda_0 = \lambda_1 = \nu + \frac{1}{2}, \lambda_2 = 0$, the transmutation $T = z^{\nu+1/2}\,I_2^{-\nu/2,v+1/2}f(z)$ is a classical E-K operator (3) and the Laplace transform generated by it, is the well-known *Meijer transformation*:

$$\mathcal{K}_{\nu}\{f(z);s\} = \int_0^{\infty}\sqrt{zs}\,K_{\nu}(zs)f(z)dz,$$

with the McDonald function $K_{\nu}(z)$.

5. Obrechkoff transform. The most interesting and still most general case of Laplace type transform (8) (before introducing (20)), is the Obrechkoff integral transform, [20]. Dimovski [3] observed and proved that its slight modification of the form

$$\mathcal{O}\{f(z);s\} = \beta\int_0^{\infty}K\left[(zs)^{\beta}\right]\ z^{\beta(\gamma_m+1)-1}\ f(z)dz \tag{21}$$

with a kernel-function (then, not yet identified by some special function)

$$K(z) = \int_0^{\infty}\cdots\int_0^{\infty}\exp\left(-u_1-\cdots-u_{m-1}-\frac{z}{u_1\ldots u_{m-1}}\right), \tag{22}$$

can be used as integral transformation, analogous to the Laplace transform, for an operational calculus for the *hyper-Bessel differential operators* of arbitrary order $m \geq 1$:

$$B=z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}z\frac{d}{dz}\ldots z^{\alpha_{m-1}}z\frac{d}{dz}z^{\alpha_m}=z^{-\beta}P_m\left(z\frac{d}{dz}\right)=z^{-\beta}\prod_{k=1}^{m}(z\frac{d}{dz}+\beta\gamma_k)\ ,\quad 0<z<\infty\,.$$

In [5], [13, Ch.3], [7], etc., a new definition for the Obrechkoff transform is proposed, representing it as a G- (therefore, also as a H-) transformation, by identification the kernel (22) as a $G^{m,0}_{m,m}$-function:

Definition 4. $$\mathcal{O}\{f(z);s\} = \beta s^{-\beta(\gamma_m+1)+1}\int_0^{\infty}G^{m,0}_{m,m}\left[(zs)^{\beta}\,\middle|\,\begin{matrix}-\\(\gamma_k+1-\frac{1}{\beta}\end{matrix}\right]f(z)dz.$$

Thus it is seen that the Obrechkoff transform is special case of (20), when $\rho_i = \beta$, $\mu_i = \gamma_i + 1, i = 1, \dots, m$, and the transmutation T is the so-called *Sonine-Dimovski transformation*: $Tf(z) = z^{\beta(\gamma_m+1)-\beta/m}\, I^{(\gamma_k),(-k/m)}_{(\beta,\dots,\beta),m-1} f(z)$. Ch. 3 of [13] illustrates how efficient can be the use of the SF (Meijer's G-function as case of the H-function), in deriving as possibly whole theory of a generalized integral transformation like the Obrechkoff one. The readers can find there condensed explicit expressions for the images of many functions, for family of convolutions, several inversion formulas, the basic operational rules. Also, many particular cases of the Obrechkoff transform (thus also of the multi-index Borel-Dzrabshjan transform (20)) are listed, on which different authors spent a lot of efforts to study, thus emphasizing the Bulgarian priority in this field.

To conclude, let us mention that a generalized integral transformation of the other mentioned type: i) Fourier-Hankel type, has been studied in analogical way, see [19]. This is the *Hankel-type integral transformation corresponding to the hyper-Bessel operators*.

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ОБОБЩЕНИЯ НА ДРОБНОТО СМЯТАНЕ, СПЕЦИАЛНИТЕ ФУНКЦИИ И ИНТЕГРАЛНИТЕ ТРАНСФОРМАЦИИ: КАКВА Е ВРЪЗКАТА?

Вирджиния С. Кирякова

В този обзор илюстрираме накратко наши приноси към обобщенията на дробното смятане (анализ) като теория на операторите за интегриране и диференциране от произволен (дробен) ред, на класическите специални функции и на интегралните трансформации от лапласов тип. Показано е, че тези три области на анализа са тясно свързани и взаимно индуцират своето възникване и по-нататъшно развитие. За конкретните твърдения, доказателства и примери, вж. Литературата.