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BRANCHING STOCHASTIC PROCESSES: HISTORY, THEORY, APPLICATIONS*

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Branching stochastic processes can be considered as models in population dynamics, where the objects have a random lifetime and reproduction following some stochastic laws. Typical examples are nuclear reactions, cell proliferation and biological reproduction, some chemical reactions, economics and financial phenomena. In this survey paper we try to present briefly some of the most important and interesting facts from the theory of branching processes and to point out some applications.

1. Brief history of branching processes. The term *Branching Process* was introduced by A. N. Kolmogorov and N. A. Dmitriev in 1947 [17] but the history starts much earlier and goes back to more than a century and a half ago in connection to the problem of the extinction of certain family lines of European aristocracy. Until 1977 [15] it was considered that the branching process was firstly used in England by F. Galton who formulated and published in 1873 in the *Educational Times* [7] the following problem:

"Problem 4001: A large nation, of whom we only concern ourselves with the adult males, N in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, p_0 percent of adult males have no male children who reach adult life; p_1 per cent have one such male child; p_2 per cent have two; and so on up to p_5 who have five.

Find (1) what proportion of surnames will have became extinct after r generations; and (2) how many instances there will be of the same surname being held by m persons."

However, in 1977, C. C. Heyde and E. Seneta (see ([13]) showed that, in France, I. J. Bienaymé had considered the family extinction problem before F. Galton published it. Indeed, they pointed out that I. J. Bienaymé was not only the first to formulate the mathematical problem, but indeed knew its solution already in 1845, although the original publication has not been found.

H. W. Watson proposed a solution to Galton's problem using the theory of generating functions and functional iteration. He established that the number of adult males with the same family name in the t-th generation is determined by the coefficients of the t-th functional iteration of the offspring generating function. Rejecting Galton's restriction $k \leq 5$ and denoting the probability generating function (p.g.f.) of the offspring of a male by $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $0 \leq s \leq 1$, then the p.g.f. of the number of adult males

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with the given family in the t-th generation is $f_t(s) = f_{t-1}(f(s))$. Watson proved that the probability q_t the family name to extinct in the $t-$th generation is: $q_1 = p_0$, $q_2 = f(q_1), \dots, q_t = f(q_{t-1})$. Assuming that the sequence q_t possesses a limit q, it has to satisfy the equation $q = f(q)$. But, this equation always has the solution $q = 1$. This made Watson to conclude erroneously that every family will die out, even when the population size, on average, increases from generation to generation (see [33]).

The problem re-appeared in Denmark and England in the period 1920-1930 with the works of R. A. Fisher, J. B. S. Haldane, K. Erlang, and J. Steffensen. Fisher [3] and Haldane [10] applied the model to some problems of genetics. In 1929 Erlang derived the equation for the probability for extinction. He proposed but did not prove that the equation has 2 solutions depending on the magnitude of $m = p_1 + 2p_2 + \dots + np_n + \dots$ (the mean number of direct descendants who inherited the given family name.) In this way he gave a correct formulation of the theorem for extinction of the process depending on the criticality. The proof was published in 1930 by J. Steffensen (see [28]). He also noted that the distribution

$$p_0 = \alpha, p_k = (1-\alpha)(1-\beta)\beta^{k-1}, \quad k = 1, 2, 3, \dots, \tag{1}$$

has fractional-linear generating function which keeps the same form in the successive iterations.

The period 1940–1950 was very important for the theory of branching processes. During this period to East and West the scientists worked on studying the nuclear chain reaction and constructing the nuclear weapon. This was a new direction for an application of the branching processes. The basic limit theorems for the BGW branching processes were proved during this period. These basic results are due to A. N. Kolmogorov, N. M. Dmitriev, A. M. Yaglom, B. A. Sevastyanov, R. Bellman, T. Harris and others (e.g. [16], [17], [34]). During this period some new models were introduced, like: multitype branching processes (Kolmogorov and others); age-dependent branching processes (Bellman-Harris model, Sevastyanov model); continuous time Markov branching processes, single and multitype.

Since then many new types of branching processes were introduced and studied, in order to get more and more adequate models to different practical problems. There were published more than 10 monographs ([11], [26], [1], [23], [14], [42], [8], [25], [9] etc.), several important surveys ([32], [29], [41] etc.) and a vast number of journal papers. A group of bulgarian mathematicians works from many years in the field of branching processes and has made significant contributions to the theory and applications. As a result in 1993 in Varna held the first world congress on branching processes in which participated the most outstanding specialists from all over the world (see [12]).

In the next sections we give a rigorous mathematical definition of the Bienemé-Galton-Watson (BGW) branching process and mention its basic properties and the directions of its generalizations. Some applications of branching processes are discussed in the last section.

2. Theory. Let us have on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ the set

$$\mathcal{X} = \{X_i(t), i = 1, 2, \dots; t = 1, 2, \dots\}$$

of independent, identically distributed (i.i.d.), nonnegative, integer valued random variables (r.v.) with distribution $p_j = \mathbf{P}\{X_k(t) = j\}$, $j = 0, 1, 2, \dots$ and probability generating function (p.g.f.) $f(s) = \sum_{j=0}^{\infty} p_j s^j$, $0 \le s \le 1$. The BGW branching process can be defined as follows

$$Z(0) = 1, \quad Z(t+1) = \sum_{k=1}^{Z(t)} X_k(t+1), \quad t = 0, 1, 2, \dots, \quad \text{where } \sum_{k=1}^{0} . = 0.$$

The process $Z(t)$ describes the evolution of the population of individuals (particles) of the same type, where $Z(t)$ is the number of particles in the t-th generation, $X_k(t+1)$ is the number of the direct descendants in the $t+1$-th generation of the k-th individual living in t-th generation. The independence of the r.v.'s $X_k(t)$ represents the basic property of the BGW branching processes, which means that every particle produces its offspring independently of the other particles in the same and in the previous generations.

The event $\{Z(t) \to 0\} = \bigcup_{t=0}^{\infty}\{Z(t) = 0\}$ means the extinction of the process. The probability $q = \lim_{t\to\infty} \mathbf{P}\{Z(t) = 0\}$ is the probability for extinction. The problem to find this probability was the essence of the Galton's problem for extinction of the family names and it is the basic problem for every branching process. A BGW branching process is called *subcritical, critical or supercritical* if $m < 1, m = 1$ or $m > 1$, respectively, where $m = f'(1) = \mathbf{E}\{X_k(t)\}$ is the mean number of offspring of a particle. We denote by $\sigma^2 = \mathbf{D}\{X_k(t)\}$ the variance of the offspring of a particle. The theorem, proved by Steffensen, asserts that the probability for extinction q is the smallest root of the equation $f(s) = s$, and $q = 1$ iff $m \le 1$, but $q < 1$ if $m > 1$.

The second basic problem is to determine the asymptotic behavior of the process when $t \to \infty$. Since $Z(t)$ cannot stay positive and bounded, i.e. $\mathbf{P}\{Z(t) \to 0 \text{ or } Z(t) \to \infty\} = 1$, the problem consists of three components:

– Asymptotic behavior of the probability for non-extinction $\mathbf{P}\{Z(t) > 0\}$. Depending on the criticality one has

$$\mathbf{P}\{Z(t) > 0\} \sim K m^t, \text{ (if } m < 1); \quad \sim \frac{2}{\sigma^2 t}, \text{(if } m = 1); \quad \to 1 - q \in (0,1), \text{(if } m > 1).$$

– Asymptotic behavior of the mathematical expectation $\mathbf{E}\{Z(t)\}$ and the variance $\mathbf{D}\{Z(t)\}$. Depending on the criticality one has for the mathematical expectation that

$$\mathbf{E}\{Z(t)\} = 1, \text{ (if } m = 1); \quad \mathbf{E}\{Z(t)\} \sim \text{const.} m^t, \quad \text{(if } m \ne 1).$$

– Finding a proper limit distribution under an appropriate normalization. Since in the subcritical case the process dies out with probability 1, a non-degenerate limit distribution could be obtained conditioning on non-extinction. So, for $k \ge 1$, $\lim_{t\to\infty} \mathbf{P}\{Z(t) = k | Z(t) > 0\} = b_k$, where $\sum_{k=1}^{\infty} b_k = 1$, $\sum_{k=1}^{\infty} k b_k < \infty$ iff $\sum_{k=1}^{\infty} k p_k \log k < \infty$. This result was proved by Yaglom [34].

In the critical case the process again dies out with probability 1, and a non-degenerate limit distribution could be obtained conditioning on non-extinction. It appears that the limits $b_k = 0$ for all $k \ge 1$. In other words the non-degenerate sample paths tend to infinity. Then, for a proper limit distribution it needs some normalization. Kolmogorov and Yaglom proved that

$$\lim_{t\to\infty} \mathbf{P}\left\{Z(t)/t > x | Z(t) > 0\right\} = e^{-x/\sigma^2}, x \ge 0,$$

i.e. the non-degenerate sample paths grow linearly.

In the three cases it is true that $W(t) = Z(t)/m^t, t \geq 0$ is a nonnegative martingale. In the subcritical and critical case $W(t) \to 0$ with probability 1, but in the supercritical case this martingale has a positive limit W. The r.v. W is such that $\mathbf{P}(W = 0) = q < 1$ and $\mathbf{P}(W > 0) = 1 - q > 0$ iff $\sum_{k=1}^{\infty} kp_k \log k < \infty$. Therefore, in the supercritical case $Z(t) \sim Wm^t, \quad t \to \infty$, i.e. the non-degenerate trajectories grow up exponentially. The proofs of these results can be found in the books of Sevastyanov [26], Athreya and Ney [1], Jagers [14] and others.

A direct generalization of the BGW branching processes are the multitype branching processes introduced by Kolmogorov [16]. They describe the evolution of particles of K different types. Every particle of a given type i lives a unit time and at the end of its life gives rise to random number of particles of all types according to a given multivariate distribution $p_j^i(k), i = 1, 2, \ldots, K; j = 1, 2, \ldots, K; k = 0, 1, 2, \ldots$. Similarly to the BGW processes it is assumed that the evolution of the particles does not depend on each other. In this case the critical property of the process is determined by the matrix of the mathematical expectations $M = ||m_j^i||_{K \times K}$, where $m_j^i = \sum_{k=0}^{\infty} kp_j^i(k)$ is the mean number of direct descendants of type j produced by a particle of type i.

The independence of the evolution of the particles of each other and the independence from the number of generation is a restriction for some applications of branching processes especially in biology. For example, the evolution of microorganisms in a closed environment depends on their concentration and on the concentration of the food in the environment. In order to get more realistic models in this direction, there were introduced new models in which the distribution of the offspring changes from generation to generation, like branching processes with density dependent evolution; branching processes in varying environment and in random environment. In many populations, mating is an important factor that can not be neglected. Bisexual branching processes take this into account explicitly. In general, these processes start with N couples. Each couple has random numbers of female and male offspring which form the next generation and so on. The bisexual branching processes are extensively studied by a group of Spanish mathematicians (see e.g. [6], Ch. 20).

The branching processes mentioned above are of discrete time. Their evolution goes from generation to generation, without taking into account the life time of the particles. The first continuous time branching process was introduced by R. Bellman and T. Harris in 1948. They supposed that every particle lives a random time with distribution function $G(t)$ and at the end of its life gives rise to its offspring like in BGW branching process. This assumption makes the evolution continuous in time. The model was generalized by Sevastyanov by the assumption that the offspring distribution depends on the age of a particle. A further generalization was made by Cramp, Mode and Jagers, assuming that a particle can bear children many times during its life and not only at the end of its life. The basic results for these models can be found in the books [26], [1], [14], [23].

In all the models described by now a basic assumption is that their evolution goes in an isolated environment and it is not possible to add or remove particles or in general to control the number of particles which will give offspring in the next generation. In 1974

B. A. Sevastyanov and A. M. Zubkov [27] introduced a class of branching processes in which it is possible to control the number of the particles in a given generation which give offspring in the next generation by a control function ϕ and called them ϕ-controlled branching processes. Let on the common probability space $(\Omega, \mathcal{A}, \mathbf{P})$ be given the set $\mathcal{X} = \{X_i(t), i = 1, 2, \ldots; t = 1, 2, \ldots\}$ of i.i.d. nonnegative, integer valued r.v. with distribution $p_j = \mathbf{P}\{X_k(t) = j\},\ j = 0, 1, 2, \ldots$ and p.g.f. $f(s) = \sum_{j=0}^{\infty} p_j s^j$, $0 \le s \le 1$, and $\Phi = \{\phi_t(k), k = 0, 1, 2, \ldots; t = 0, 1, 2, , \ldots\}$ where for each $t = 0, 1, 2, \ldots,\ \{\phi_t(k)\}_{k \ge 0}$ are independent integer valued stochastic processes with equal one dimensional distributions. Then, following N. M. Yanev [35] a ϕ-controlled branching process with random control functions is defined as follows

$$Z(0) = N > 0, Z(t+1) = \sum_{i=0}^{\phi_t(Z(t))} X_i(t+1), t = 0, 1, 2, \ldots.$$

The model of Sevastyanov and Zubkov assumes non random control function, the same for the all generations. Some classes of branching processes studied earlier can be viewed as special cases of controlled branching processes. These are:

– branching processes with immigration, i.e. the possibility to add new particles from outside of the process (in each or in some generations). (Most of the branching processes have been studied also with an immigration component (e.g. [14]));

– branching processes with emigration, i.e. removing particles from a given generation and they do not give offspring in the next generation. (The model was studied for BGW processes by Vatutin [30]);

– branching processes allowing random migration (immigration or emigration of particles in each generation.

The last model was introduced by S. V. Nagaev and L. V. Khan [24] and N. M. Yanev and K. V. Mitov [36] in 1980 independently, as a special case of ϕ controlled branching processes. On the other hand, it is enough general to include all types of discrete time Markov chains. The model was extensively studied by N. M. Yanev and G.P. Yanev in [37], [38], [39], [40]. The definition below follows [40]. Denote by $U_t(j) = \sum_{i=1}^{j} X_i(t)$, $j = 1, 2, \ldots,$ and $U_t(0) \equiv 0$. Assume that on the same probability space there are two mutually independent and independent of $\mathcal{X}$ sequences $E = \{e_1(t), e_2(t)\}$ and $I = \{I_t^+, I_t^o\}$, each of which consists of independent random vectors with nonnegative integer valued components with p.g.f. $H(s_1, s_2) = \mathbf{E}\{s_1^{e_1(t)} s_2^{e_2(t)}\}$ and $G(s_1, s_2) = \mathbf{E}\{s_1^{I_t^+} s_2^{I_t^o}\}$. The process with random migration $\{Z(t), t = 0, 1, 2, \ldots\}$ is defined as follows:

$$Z(0) = 0,\ Z(t) = (U_t(Z(t-1)) + M_t)^+,\ t = 1, 2, \ldots, \tag{2}$$

where $\mathbf{P}\{M_t = -(U_t(e_1(t)) + e_2(t))\} = p$, $\mathbf{P}\{M_t = 0\} = q$, and $\mathbf{P}\{M_t = I_t^+ \mathbb{I}_{\{Z(t-1)>0\}} + I_t^o \mathbb{I}_{\{Z(t-1)=0\}}\} = r$, and $p, q, r \ge 0, \quad p + q + r = 1$. As usually $a^+ = \max\{a, 0\}$. If $Z(t-1)$ is the number of particles in the $(t-1)$-st generation. Then we have three options for the t-th generation:

(i) With probability p, $e_1(t)$ families emigrate, and their $U_t(e_1(t))$ children not go in the t-th generation (family emigration) and also $e_2(t)$ randomly selected individuals are removed from the t-th generation (individual emigration);

(ii) There is no any migration with probability q, i.e. the process evolves like BGW process;

(iii) With probability r, I_t^+ new particles immigrate if $(t-1)$-st generation is non-empty or I_t^o new individuals immigrate if $(t-1)$-st generation is empty.

If $q=1$, then $\{Z(t), t=0,1,2,\ldots\}$ is a classical BGW process. If $r=1$ and $I_t^+ \equiv I_t^o$ one gets the classical BGW process with immigration. If $r=1$ and $I_t^+ \equiv 0$, then the process is the Foster-Pakes process with state-dependent immigration. The case when $p=1$, $e_1(t) \equiv 1$ and $e_2(t) \equiv 0$ is the process of pure emigration [30].

Assuming that the process is critical with finite variance, i.e. $m = \mathbf{E}\{X_i(t)\} = 1$ and $0 < \mathbf{D}\{X_i(t)\} = \sigma^2 < \infty$, and the migration satisfies the following conditions $0 < m_I^+ = \mathbf{E}\{I_t^+\} < \infty$, $0 < m_E = \mathbf{E}\{e_1(t) + e_2(t)\} < \infty$, $0 < m_I^o = \mathbf{E}\{I_t^o\} < \infty$, $0 \le e_1(t) \le N_1 < \infty$, $0 \le e_2(t) \le N_2 < \infty$, *a.s.* N. M. Yanev and G. P. Yanev ([40]) proved that the parameter $\Delta = \dfrac{rm_I^+ - pm_E}{\sigma^2/2}$ plays a crucial role for the limiting behavior of the process: (i) If $\Delta < 0$, then the process possesses a non-degenerate stationary distribution; (ii) if $\Delta = 0$, then $\lim\limits_{t\to\infty} \mathbf{P}\{\log Z(t)/\log t \le x\} = x$ for $0 < x < 1$; (iii) if $\Delta > 0$, then $\lim\limits_{t\to\infty} \mathbf{P}\left\{2Z(t)/\sigma^2 t \le x\right\} = \Gamma(\Delta)^{-1} \displaystyle\int_0^x y^{\Delta-1} e^{-y} dy$ for $x \ge 0$. Together with the process $Z(t)$ they studied in details the process $Z_1(t)$ which coincides with $Z(t)$ until its first visit of the state zero after beginning.

An interesting idea for controlled branching processes is to control a branching process by means of another branching process. It is developed by P. Mayster [19].

In this paper we do not pay any attention to the regenerative branching processes which are studied by Bulgarian mathematicians. The survey [20] is devoted to these processes. The results of Bulgarian mathematicians till 2008 are comprehensively represented in [22].

3. Applications. A. J. Lotka had seen the Steffensen's work and immediately (see [18]) applied the branching process theorem to the data contained in the 1920 United States census of white males, obtaining $q = 0,88$ as the probability of the termination of the male line of descent from newly born male. He also made the observation that for the population under study the probability distribution (1) fits the facts with $\alpha = 0.4813$ and $\beta = 0.5586$. For this distribution $q = 0.862$ and $m = \dfrac{1-\alpha}{1-\beta} = 1.175 > 1$.

In spite of its simplicity the BGW branching processes continue to get new applications. They are used as models of: the initial period of the spread of epidemic disease ([5], [4]); the time to populate an environment [2]; stock and option pricing [21]; phenomena in nuclear reactors and photon-neutron cascades [31] and others.

More complex branching processes find new and new applications to real world problems. The book of A. Yu. Yakovlev and N. M. Yanev [42] contains applications of different types of branching processes to cell proliferation kinetics. Yakovlev, Yanev and their collaborators continued successfully to work in this direction during the last decade. As a result age-dependent branching processes, both single and multitype were used for modeling of the kinetics of cancer cells. A new age-dependent branching process with non-homogeneous Poisson immigration was studied. These and other applications of branching processes in molecular biology are presented in [45]. Chapters 12–18 of [6] also contain many new and interesting ideas for applications of branching processes in cell proliferation, genetics, epidemiology and many other areas of science and practice.

The applications of branching processes to real world problems requested estimations of their parameters. So, the statistics of branching was developing during the last three decades. One of the first estimators for the mathematical expectation of the offspring of a particle is due to Lotka. Later on many peoples were involved in this area. The book of P. Guttorp [8] is especially devoted to the statistics of branching processes.

Bulgarian mathematicians also have made an important contribution to the statistical inference of branching processes. A very good and comprehensive survey of the resent results in this area is [41]. Recently there were obtained robust estimators for the individual probabilities, mathematical expectation and variance of the offspring distribution of a particle (e.g. [43]). The robust estimators of the offspring mean are obtained also for BGW processes with immigration [44].

Unfortunately, in this brief discussion it is not possible to mention even a small part of the theory and applications of branching processes, but we hope that the talk will increase the popularity of this interesting and important area of probability theory.

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РАЗКЛОНЯВАЩИ СЕ СТОХАСТИЧНИ ПРОЦЕСИ: ИСТОРИЯ, ТЕОРИЯ, ПРИЛОЖЕНИЯ

Косто В. Митов

Разклоняващите се стохастични процеси са модели на популационната динамика на обекти, които имат случайно време на живот и произвеждат потомци в съответствие с дадени вероятностни закони. Типични примери са ядрените реакции, клетъчната пролиферация, биологичното размножаване, някои химични реакции, икономически и финансови явления. В този обзор сме се опитали да представим съвсем накратко някои от най-важните моменти и факти от историята, теорията и приложенията на разклоняващите се процеси.