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ON SOME JACOBI SERIES*

Georgi Boychev

The paper presents properties of some Jacobi series.

Suppose that $\alpha+1$, $\beta+1$ and $\alpha+\beta+2$ are not equal to $0, -1, -2, \ldots$. The polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ defined by equalities

$$P_n^{(\alpha,\beta)}(z)=\binom{n+\alpha}{n}F\left(-n,n+\alpha+\beta+1,\alpha+1;\frac{1-z}{2}\right),\quad n=0,1,2,\ldots;\ z\in\mathbb{C},$$

where $\mathbb{C}$ is the complex plane and $F(a,b,c;\zeta)$ is Gauss hypergeometric function, are called Jacobi polynomials with parameters α and β. The functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ defined by equalities

$$Q_n^{(\alpha,\beta)}(z)=\frac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(z-1)^{n+1}}F\left(n,n+\alpha+1,2n+\alpha+\beta+2;\frac{2}{1-z}\right),$$

$$n=0,1,2,\ldots;\quad z\in G=\mathbb{C}\setminus[-1,1],$$

are called Jacobi associated functions.

Let $\omega(z)$ be that inverse of Zhukovskii function in the region G for which $|\omega(z)|>1$. Then, in the region G the Jacobi polynomials and Jacobi associated functions have respectively the representations $(n\ge 1)$ [1, Chapter III, (1.9), (1.30)]

$$P_n^{(\alpha,\beta)}(z)=P^{(\alpha,\beta)}(z)n^{-\frac{1}{2}}[\omega(z)]^n\{1+p_n^{(\alpha,\beta)}(z)\},\tag{1}$$

and

$$Q_n^{(\alpha,\beta)}(z)=Q^{(\alpha,\beta)}(z)n^{-\frac{1}{2}}[\omega(z)]^{-n-1}\{1+q_n^{(\alpha,\beta)}(z)\}\tag{2}$$

where $P^{(\alpha,\beta)}(z)\neq 0$, $Q^{(\alpha,\beta)}(z)\neq 0$,$\{p_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$, and $\{q_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$ are holomorhpic functions in the region G.

If $n\to+\infty$, then

$$p_n^{(\alpha,\beta)}(z)=O(n^{-1})\tag{3}$$

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and

(4) $$q_n^{(\alpha,\beta)}(z) = O(n^{-1})$$

uniformly on every compact subset of G.

We call the series of the kind

(5) $$\sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z)$$

Jacobi series.

If

$$0 < r^{-1} = \lim_{n\to+\infty} \sup |a_n|^{\frac{1}{n}} < 1,$$

then the series (5) is absolutely and uniformly convergent on every compact subset of the region $E(r) = \{z \in \mathbb{C} : |z+1| + |z-1| < r + r^{-1}\}$ and divergent in $\mathbb{C}\backslash\overline{E(r)}$ [1, (IV.1.1),(b)]. Let $\gamma(r) = \partial E(r)$ for $r > 1$.

Theorem 1 [1, (V.1.3)]**.** *Let $f(z)$ be a complex function holomorphic in $E(R)$, where $R > 1$. Then, the function $f(z)$ is representable in $E(R)$ by a series of the kind* (1)*, i.e. $f(z) = \sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z)$, $z \in E(R)$, with coefficients*

$$a_n = \frac{1}{2i\pi I_n^{(\alpha,\beta)}} \int_{\gamma(r)} f(\varsigma) Q_n^{(\alpha,\beta)}(\varsigma) d\varsigma, 1 < r < R, n = 0, 1, 2, \ldots,$$

where

$$I_n^{(\alpha,\beta)} = \begin{cases} \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}, & n = 0 \\ \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n++\alpha+\beta+1)}, & n \geq 1 \end{cases}.$$

Now we shall prove the following

Theorem 2. *Let $1 < R < +\infty$, $\alpha, \beta, \alpha+\beta+1 \neq -1, -2, \ldots$ and $f(z)$ be a complex function holomorphic and bounded in the region $E(R)$. Let $\{S_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ be the partial sums of Jacobi's series, representing the function $f(z)$ in $E(R)$. Then,*

(6) $$S_n^{(\alpha,\beta)}(z) = O(\ln n), \quad n \to +\infty, \quad z \in E(R).$$

Proof. Let M be a constant for which

(7) $$|f(z)| \leq M, \quad z \in E(R).$$

We assume that $r \in \Delta(R) = \left[\frac{R+1}{2}, R\right)$.

Using (1) and (2) it is easy to prove that

$$S_n^{(\alpha,\beta)}(z) = \frac{1}{2\pi i}\int\limits_{\gamma(r)} \frac{1-[\omega(z)/\omega(\varsigma)]^n}{\varsigma - z} f(\varsigma)d\varsigma$$
$$+\frac{1}{2\pi i}\int\limits_{\gamma(r)} \frac{D^{(\alpha,\beta)}(\varsigma,\varsigma)-D^{(\alpha,\beta)}(z,\varsigma)}{\varsigma - z}[\omega(z)/\omega(\varsigma)]^n f(\varsigma)d\varsigma$$
$$-\frac{1}{2\pi i}\int\limits_{\gamma(r)} \frac{\delta_n^{(\alpha,\beta)}(z,\varsigma)}{\varsigma - z}[\omega(z)/\omega(\varsigma)]^n f(\varsigma)d\varsigma = J_{n,1}+J_{n,2}-J_{n,3},$$

where $D^{(\alpha,\beta)}(z,\varsigma)$ and $\{\delta_n^{(\alpha,\beta)}(z,\varsigma)\}_{n=1}^{+\infty}$ are complex-valued functions holomorphic in the region $G\times G$. Moreover, $D^{(\alpha,\beta)}(z,z)\equiv 1$ and $\delta_n^{(\alpha,\beta)}(z,z)\equiv 0$ $(n=1,2,\dots)$ in G.

Using (3) and (4) it is not difficult to prove that

$$(z-\varsigma)\delta_n^{(\alpha,\beta)}(z,\varsigma) = O(n^{-1})(n\to+\infty)$$

uniformly on every compact subset of $G\times G$. Then we have

$$|J_{n,3}| \le K_1 n^{-1}\int\limits_{\gamma(r)} |f(\varsigma)||d\varsigma| \le K_1 n^{-1} M \int\limits_{\gamma(r)} |d\varsigma| \le K_2 n^{-1},$$

where K_1 and K_2 are constants, which do not depend on r and n. Hence,

$$J_{n,3} = O(n^{-1})(n\to+\infty) \tag{8}$$

uniformly with respect to $r\in\Delta(R)$.

It is easy to prove that for $|\omega(z)|,|\omega(\zeta)|\in\Delta(R)$, we have that

$$\left|\frac{D^{(\alpha,\beta)}(\varsigma,\varsigma)-D^{(\alpha,\beta)}(z,\varsigma)}{\varsigma - z}\right| \le K_3,$$

where K_3 is constant. Then,

$$|J_{n,2}| \le \frac{1}{2\pi}K_3\int\limits_{\gamma(r)} |f(\varsigma)||d\varsigma| \le \frac{1}{2\pi}K_3 M\int\limits_{\gamma(r)} |d\varsigma| \le rK_3M \le RK_3M.$$

From this inequality it follows that

$$J_{n,2} = O(1) \quad (n\to+\infty) \tag{9}$$

uniformly with respect to $r\in\Delta(R)$.

For the integral $J_{n,1}$ we have the representation

$$J_{n,1} = \frac{1}{2\pi i}\int\limits_{\gamma(r)} \frac{\omega(\varsigma)-\omega(z)}{\varsigma - z}\,\frac{1-[\omega(z)/\omega(\varsigma)]^n}{\omega(\varsigma)-\omega(z)} f(\varsigma)d\varsigma.$$

Obviously the function $[\omega(\varsigma)-\omega(z)]/(\varsigma-z)$ is bounded for $|\omega(\varsigma)|,|\omega(z)|\in\Delta(R)$. Let $F(\zeta,z) = \dfrac{\omega(\varsigma)-\omega(z)}{\varsigma - z}f(z)$. Then, using (7) we get that $|F(\zeta,z)|\le K_4$, where K_4 is a constant.

Let $\omega(z) = r\exp i\theta$, where $\theta \in [-\pi, \pi]$ and $r \in \Delta(R)$. Putting $\omega(\zeta) = r\exp i\tau$ $(\tau \in [-\pi+\theta, \pi+\theta], r \in \Delta(R))$, we obtain that

$$J_{n,1} = \frac{1}{2\pi}\int\limits_{-\pi+\theta}^{\pi+\theta} F_1(\tau,\theta)\frac{1-\exp(in(\tau-\theta))}{1-\exp(i(\tau-\theta))}(1-r^{-2}\exp(-2i\tau))d\tau,$$

where $F_1(\tau,\theta) = F[(\omega(\zeta)+\omega^{-1}(\zeta))/2, (\omega(z)+\omega^{-1}(z))/2]$ is a periodical function with respect to τ and θ. Using substitution $t = \theta - \tau$ in integral $J_{n,1}$ we get

$$J_{n,1} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} F_1(\theta-\tau,\theta)\frac{1-\exp nti}{1-\exp ti}[r^{-2}\exp 2i(t-\tau)-1]dt.$$

Obviously, $|F_1(\tau,\theta)| \leq K_4$. Then, using the inequality $r > 1$ we obtain that

$$|J_{n,1}| \leq K_5\int\limits_{-\pi}^{\pi}\frac{|\sin(nt/2)|}{|\sin(t/2)|}dt \leq K_6\int\limits_{0}^{\pi/2}\frac{|\sin nu|}{\sin u}du,$$

where K_5 and K_6 are constants.

Let $I = \int_0^{\pi/2}\frac{|\sin nu|}{\sin u}du$. Then $I = \int_0^{1/n}\frac{\sin nu}{\sin u}du + \int\limits_{1/n}^{\pi/2}\frac{|\sin nu|}{\sin u}du = I_1 + I_2$.

Using the inequality $|\sin nu| \leq n\sin u$we obtain that $I_1 \leq n\int_0^{1/n} du = 1$. Therefore,

$$I_1 = O(1)(n \to +\infty). \tag{10}$$

Using that $\sin u \geq 2u/\pi$ for $u \in (0, \pi/2)$, we get

$$I_2 \leq \frac{\pi}{2}\int_{1/n}^{\pi/2}\frac{|\sin nu|}{u}du \leq \frac{\pi}{2}\int_{1/n}^{\pi/2}\frac{1}{u}du = \frac{\pi}{2}\left(\ln\frac{\pi}{2} - \ln\frac{1}{n}\right) = \frac{\pi}{2}\left(\ln\frac{\pi}{2} + \ln n\right).$$

Hence,

$$I_2 = O(\ln n)(n \to +\infty). \tag{11}$$

From (10) and (11) it follows that

$$J_{n,1} = O(\ln n)(n \to +\infty). \tag{12}$$

Using asymptotic formulas (12), (9) and (8), we get the asymptotic formula (6) for these z for which $|\omega(z)| \in \Delta(R)$. Then, it is not difficult to prove that (6) is valid for every $z \in E(R)$. Thus Theorem 2 is proved. □

As a corollary of Theorem 2 we can state the following proposition:

Theorem 3. *Let* $1 < R < +\infty$, $\alpha, \beta, \alpha+\beta+1 \neq -1, -2, \ldots$ *and* $f(z)$ *be a complex function holomorphic and bounded in the region* $E(R)$. *Let* $\{S_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ *be the partial* *sums of the Jacobi series, representing the function $f(z)$ in $E(R)$. If*

$$\sigma_n^{(\alpha,\beta)}(z) = \frac{1}{n+1}\sum_{j=0}^{n} S_j^{(\alpha,\beta}(z) \quad (n = 0, 1, 2, \dots),$$

then $\{\sigma_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ are bounded in the region $E(R)$.

Conversely, if $\{\sigma_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ are bounded in the region $E(R)$, then $f(z)$ is bounded in $E(R)$.

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Georgi S. Boychev
42, Irechek Str.
6000 Stara Zagora, Bulgaria
e-mail: GBoychev@hotmail.com

ВЪРХУ НЯКОИ РЕДОВЕ НА ЯКОБИ

Георги С. Бойчев

Настоящата статия съдържа свойства на някои редове на Якоби.