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NOTES ON THE SOLUTION OF DYNAMIC OPTIMIZATION PROBLEMS WITH EXPLICIT CONTROLS IN DISCRETE-TIME ECONOMIC MODELS*

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In the paper methods of solving discrete-time infinite-horizon dynamic optimization problems with explicit controls are studied. It is provided a justification of a solution procedure based on a Lagrangian formulation that is frequently applied to such problems in the economics literature. Necessary conditions for optimality are derived on the basis of the Bellman equation and sufficient conditions for optimality are provided under assumptions commonly employed in economics.

1. Introduction. Dynamic optimization problems appear regularly in economic models. One frequently encountered formulation takes the form of an infinite-horizon optimal control problem in discrete time. It is typical in the economics literature to either provide a solution directly, without reference to the method used to derive it (see e.g. [4]), or to use a solution recipe based on an extension of the method of Lagrange multipliers as in [3]. The present work suggests one possible way of justifying the use of the Lagrangian method to derive necessary conditions for optimality in this context and formulates a sufficiency result for optimality.

Let $X \subset \mathbb{R}^n$ be a given set (describing the state variables $x = (x^1, \dots, x^n)$.) We assume that for any $x \in X$ there is a set $U(x) \subset \mathbb{R}^m$, whose elements $u = (u^1, \dots, u^m)$ are called controls. With the aid of a scalar function $g(x, u)$ and a vector function $f(x, u)$ taking values in X, where $x \in X$, $u \in U(x)$, one defines a typical problem of the type considered here:

Find a sequence of admissible controls $\{u_t\}$, $t = 0, 1, 2, \dots$, for initial data x_0, which determines through the state equations

$$x_{t+1} = f(x_t, u_t) \tag{1}$$

a sequence of values for the state variables $\{x_{t+1}\}$, $t = 0, 1, 2, \dots$, for which the objective functional

$$j(x_0, \mathbf{u}) = \sum_{t=0}^{\infty} \alpha^t g(x_t, u_t) \tag{2}$$

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attains its maximum

$$J(x_0) = \sup j(x_0, \mathbf{u}), \tag{3}$$

where the sup operation is taken over the set of all admissible choices of $\{u_t\}$. We refer to this formulation as *problem A*.

The scalar $\alpha \in (0, 1)$ is called a discount factor. We denote by $\mathrm{FC}(x_0)$ the set of all feasible control sequences $\{u_t\}_{t=0}^{\infty}$ for initial data $x_0 \in X$, i.e. x_{t+1} satisfies (1) for $u_t \in U(x_t)$, $t = 0, 1, 2, \dots$, and a given x_0. We write $\{x_{t+1}^*, u_t^*\}$ $t = 0, 1, 2, \dots$, to denote the optimal sequence of state-control pairs for problem A, i.e. $\{u_t^*\} \in \mathrm{FC}(x_0)$, and

$$J(x_0) = j(x_0, \mathbf{u}^*), \ \mathbf{u}^* = \{u_t^*\}.$$

We note that convergence in (2) is guaranteed by imposing additional assumptions, e.g. boundedness of $g(x, u)$ for all feasible values of the variables.

The usual approach to problem A in the economics literature is to proceed by analogy to the finite-dimensional case and set up a Lagrangian function, which in the proposed notation has the form

$$\mathcal{L}(x_1, x_2, \dots, u_0, u_1, \dots) = \sum_{t=0}^{\infty} \alpha^t g(x_t, u_t) + \alpha^t \lambda_t \cdot [f(x_t, u_t) - x_{t+1}], \tag{4}$$

where $\lambda_t = (\lambda_t^1, \dots, \lambda_t^n)$, $t = 0, 1, 2, \dots$, are the Lagrange multipliers and

$$\lambda_t \cdot [f(x_t, u_t) - x_{t+1}] = \sum_{i=1}^{n} \lambda_t^i [f^i(x_t, u_t) - x_{t+1}^i].$$

Under the assumption of differentiability of the respective objects, one can formally differentiate $\mathcal{L}$ with respect to the coordinates of x_t and u_t to obtain the so called first-order optimality conditions:

$$\begin{aligned} &\alpha \left[g_{x_t^k}(x_t, u_t) + \sum_{i=1}^{n} \lambda_t^i f^i_{x_t^k}(x_t, u_t) \right] = \lambda_{t-1}^k, \quad k = 1, \dots, n, \\ &g_{u_t^j}(x_t, u_t) + \sum_{i=1}^{n} \lambda_t^i f^i_{u_t^j}(x_t, u_t) = 0, \quad j = 1, \dots, m. \end{aligned} \tag{5}$$

Then, combining (1) and (5), one computes the solution $\{u_t\}_{t=0}^{\infty}$ or, more precisely, the sequence $\{x_{t+1}, u_t\}_{t=0}^{\infty}$. (It is common practice to find a stationary point of the system (1) and (5) and study a linearized version of the system around that point. This approach is especially prevalent when analyzing stochastic versions of problem A, which is beyond the scope of the present work.)

It is natural to ask whether: 1) the above procedure for computing necessary conditions for optimality has a formal justification and 2) the solution $\{x_{t+1}, u_t\}_{t=0}^{\infty}$ obtained in this manner is optimal indeed.

The present work aims to provide answers to these questions in the spirit of the analysis in [5], where a similar problem is studied without the inclusion of explicit controls in the formulation.

2. Necessary conditions for optimality as a consequence of the Bellman equation. A basic fact in the study of problem A is that the function $J(x_0)$, called value function of the problem, satisfies the Bellman equation (see e.g. [1], p.4), which in the proposed notation has the form

$$J(x) = \sup_{u \in U(x)} \left[g(x,u) + \alpha J(f(x,u)) \right]. \tag{6}$$

The analysis in [5] focuses mainly on the relation between the functional equation (6) and problem A, rather than on the way in which one obtains equation (6). A relatively simple system of conditions ensures the existence of a fixed point of the operator

$$T(h)(x) = \sup_{u \in U(x)} \left[g(x,u) + \alpha h(f(x,u)) \right], \tag{7}$$

where $h \in C(X)$ and $T : C(X) \to C(X)$. This fixed point turns out to be a solution of (6).

As the operator T preserves certain properties of h, it follows that the solution $J(x)$ also possess them. In this way one establishes that if $g(x,u)$ and $f(x,u)$ are strictly increasing with respect to the x-variables (with $x \le x'$ meaning $x^k \le x'^k$, $\forall k = 1, \ldots, n$ and a strict inequality holding for at least one coordinate) and if $U(x) \subseteq U(x')$ for $x \le x'$, then $J(x)$ is strictly increasing.

As a consequence of this property we obtain that if $J(x)$ is differentiable, then its derivatives are nonnegative.

Let the supremum in (6) be attained at a point in the interior of $U(x)$. Denoting by $u = \nu(x)$ the point in $U(x)$ for which the supremum in (6) is attained and assuming that all the derivatives considered below exist, we find sequentially

$$J(x) = g(x, \nu(x)) + \alpha J(f(x, \nu(x))) \tag{8}$$

and

$$g_{u^j}(x, \nu(x)) + \alpha \sum_{i=1}^{n} \left[J_{x^i}(f(x, \nu(x))) f^i_{u^j}(x, \nu(x)) \right] = 0, \quad j = 1, \ldots, m. \tag{9}$$

Differentiating (8) with respect to x^k, $k = 1, \ldots, n$, we obtain

$$\begin{aligned} J_{x^k}(x) = & g_{x^k}(x, \nu(x)) + \sum_{j=1}^{m} g_{u^j}(x, \nu(x)) \nu^j_{x^k}(x) + \\ & \alpha \sum_{i=1}^{n} J_{x^i}(f(x, \nu(x))) \left[f^i_{x^k}(x, \nu(x)) + \sum_{j=1}^{m} f^i_{u^j}(x, \nu(x)) \nu^j_{x^k} \right]. \end{aligned} \tag{10}$$

In view of (9), we have

$$J_{x^k}(x) = g_{x^k}(x, \nu(x)) + \alpha \sum_{i=1}^{n} J_{x^i}(f(x, \nu(x))) f^i_{x^k}(x, \nu(x)). \tag{11}$$

Let us denote $u^*_t = \nu(x^*_t)$ for $t = 0, 1, 2, \ldots$, with x_0 given and x^*_{t+1}, $t \ge 0$, determined by (1). In view of the counterparts of theorems 4.4 and 4.5 in [5] for the case considered here, under a supplementary condition on the solution $J(x)$ to (6) of the form

$$\lim_{t \to \infty} \alpha^t J(x^*_t) = 0, \tag{12}$$

the sequence $\{x^*_{t+1}, u^*_t\}$, $t = 0, 1, 2, \ldots$, is a solution to problem A. Taking the points

$x = x_t^*$ and $u_t^* = \nu(x_t^*)$ in (9) and (11), we obtain

$$g_{u^j}(x_t^*, u_t^*) + \alpha \sum_{i=1}^{n} \left[J_{x^i}(x_{t+1}^*) f_{u^j}^i(x_t^*, u_t^*) \right] = 0, \quad j = 1, \ldots, m, \tag{13}$$

and

$$J_{x^k}(x_t^*) = g_{x^k}(x_t^*, u_t^*) + \alpha \sum_{i=1}^{n} J_{x^i}(x_{t+1}^*) f_{x^k}^i(x_t^*, u_t^*). \tag{14}$$

The last two conditions coincide with those in (5) if we set

$$\lambda_t = \alpha \operatorname{grad} J(x_{t+1}^*). \tag{15}$$

We can summarize the considerations presented above as follows:

Theorem 1. *Let $\{x_{t+1}^*, u_t^*\}_{t=0}^{\infty}$ be a solution to problem A. Define λ_t as in* (15) *and assume that the following conditions hold:*

1. *$g(x, u)$ and $f(x, u)$ are strictly increasing in x;*
2. *$g(x, u)$ and $f(x, u)$ are differentiable in x and u;*
3. *$J(x)$ is differentiable in x;*
4. *$x_t^* \in \operatorname{int} X$ and $u_t^* \in \operatorname{int} U(x_t^*)$, $\forall t$.*

Then, the sequence $\{x_{t+1}^*, u_t^*\}_{t=0}^{\infty}$ satisfies the system (5).

The above provides a (partial) justification of the necessary conditions in (5) in the sense that, at least under certain conditions, there exist Lagrange multipliers λ_t for which these hold. The conditions on g and f in Theorem 1 have to be strengthened if one wants to derive some of the hypotheses of the theorem (e.g. differentiability of J) instead of assuming them.

Additionally, the representation in (15) implies that λ_t^i, $i = 1, \ldots, n$, are nonnegative (because of the monotonicity of J). This proves to be useful in establishing sufficiency and is also quite natural in view of the economic interpretation of Lagrange multipliers.

Remark 1. The derivation based on dynamic programming used in this section is potentially useful in that it could be generalized to a stochastic version of problem A. Nonetheless, the necessary conditions in (5) can also be derived by other means, for instance, by applying an appropriate version of Pontryagin's maximum principle for discrete time problems as formulated in [2]. Given a solution $\{x_{t+1}^*, u_t^*\}_{t=0}^{\infty}$ to problem A, assume for every $t = 0, 1, 2, \ldots$ that the functions f and g are differentiable at (x_t^*, u_t^*), that $u_t^* \in \operatorname{int} U(x_t^*)$ and that $f_u(x_t^*, u_t^*)$ is onto. Define the Hamiltonian for the problem as

$$H(x, u, \lambda^0, p, t) := \lambda^0 \alpha^t g(x, u) + p \cdot f(x, u),$$

where $\lambda^0 \in \mathbb{R}$ and $p \in \mathbb{R}^n$. Then, Theorem 4 in [2] states that there exist co-state variables $\{p_t\}_{t=0}^{\infty}$ such that $p_{t-1} = H_{x_t}(x_t^*, u_t^*, 1, p_t, t)$, $t \geq 1$, and $H_{u_t}(x_t^*, u_t^*, 1, p_t, t) = 0$, $t \geq 0$. Applying these to the Hamiltonian and setting $\lambda_t := p_t/\alpha^t$, we yield our equations (5).

3. Sufficient conditions for optimality. Under certain additional assumptions one can show that, analogously to Theorem 4.15 in [5], the necessary conditions obtained above are also sufficient.

Theorem 2. *Let* $\{\lambda_t\}$, $\{x^*_{t+1}, u^*_t\}$, $t = 0, 1, 2, \ldots,$ *be determined through equations* (1) *and* (5). *If*

1. *the functions* $g(x, u)$ *and* $f(x, u)$ *are concave with respect to* (x, u)*;*
2. *the Lagrange multipliers* $\lambda^1_t, \ldots, \lambda^n_t$, $t = 0, 1, 2, \ldots$ *are nonnegative;*
3. *the state space* X *is a subset of* $\mathbb{R}^n_+$ *and the transversality condition*

$$\lim_{T\to\infty} \alpha^T \lambda_T x^*_{T+1} = 0$$

holds, then, the sequence $\{x^*_{t+1}, u^*_t\}$ *(with* x_0 *given) is optimal for problem A.*

Proof. For $T \in \mathbb{N}$ and an arbitrary sequence $u_t \in \mathrm{FC}(x_0)$ (and the corresponding sequence of state variables x_t), we write

$$\mathcal{L}_T(x_t, u_t) = \sum_{t=0}^{T} \alpha^t \left\{g(x_t, u_t) + \lambda_t \cdot [f(x_t, u_t) - x_{t+1}]\right\}.$$

Then,

$$\begin{aligned}\mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x^*_t, u^*_t) = \sum_{t=0}^{T} \alpha^t \lambda_t \cdot (x^*_{t+1} - x_{t+1}) + \\ \sum_{t=0}^{T} \alpha^t [g(x_t, u_t) + \lambda_t \cdot f(x_t, u_t) - g(x^*_t, u^*_t) - \lambda_t \cdot f(x^*_t, u^*_t)],\end{aligned} \tag{16}$$

which, taking concavity into account, does not exceed

$$\begin{aligned}&\sum_{t=0}^{T} \alpha^t \lambda_t \cdot (x^*_{t+1} - x_{t+1}) + \sum_{t=0}^{T} \alpha^t \Bigg[\sum_{k=1}^{n} g_{x^k}(x^*_t, u^*_t) \cdot (x^k_t - x^{*k}_t) + \\ &\sum_{j=1}^{m} g_{u^j}(x^*_t, u^*_t) \cdot (u^j_t - u^{*j}_t) + \sum_{k=1}^{n} \lambda^k_t \Bigg\{\sum_{s=1}^{n} f^k_{x^s}(x^*_t, u^*_t) \cdot (x^s_t - x^{*s}_t) + \\ &\sum_{j=1}^{m} f^k_{u^j}(x^*_t, u^*_t) \cdot (u^j_t - u^{*j}_t)\Bigg\}\Bigg].\end{aligned}$$

Using the conditions (5), we obtain

$$\mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x^*_t, u^*_t) \le \sum_{t=0}^{T} \alpha^t \lambda_t \cdot (x^*_{t+1} - x_{t+1}) + \sum_{t=0}^{T} \alpha^t \left[\frac{\lambda_{t-1}}{\alpha} \cdot (x_t - x^*_t)\right].$$

In the last sum $x_0 = x^*_0$ and, replacing $t - 1$ with t, we find that (16) does not exceed

$$\alpha^T \lambda_T (x^*_{T+1} - x_{T+1}) \le \alpha^T \lambda_T x^*_{T+1},$$

which tends to zero by virtue of the transversality condition. □

Remark 2. The transversality condition and the assumption of convergence of the series (2) are related. They would both be true if we can guarantee that the values of the state variables and the controls 1) either lie in compact sets or 2) are *not* optimal outside a given compact.

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БЕЛЕЖКИ ВЪРХУ РЕШАВАНЕТО НА ДИМАНИЧНИ ОПТИМИЗАЦИОННИ ЗАДАЧИ С ЯВНИ УПРАВЛЕНИЯ В ИКОНОМИЧЕСКИ МОДЕЛИ С ДИСКРЕТНО ВРЕМЕ

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В работата се изследват методи за решаването на задачи на оптималното управление в дискретно време с безкраен хоризонт и явни управления. Дадена е обосновка на една процедура за решаване на такива задачи, базирана на множители на Лагранж, която често се употребява в икономическата литература. Изведени са необходимите условия за оптималност на базата на уравнения на Белман и са приведени достатъчни условия за оптималност при допускания, които често се използват в икономиката.