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SINGULAR SOLUTIONS WITH EXPONENTIAL GROWTH FOR THE (3+1)-D WAVE EQUATION*

Nedyu Popivanov, Todor Popov, Rudolf Scherer

Four-dimensional boundary value problems for the nonhomogeneous wave equation are studied. They were proposed by M. Protter as multidimensional analogues of Darboux problems in the plane. It is known that the unique generalized solution may have a strong power-type singularity at only one boundary point. This singularity is isolated at the vertex of the characteristic cone and does not propagate along the cone. Another aspect is that the problem is not Fredholm, since it has infinite-dimensional cokernel. Some known results suggest that the solution may have at most exponential growth, but the question whether such solutions really exist was still open. We show that the answer is positive and construct generalized solution of Protter problem with exponential singularity.

1. Introduction. We study boundary value problems for the wave equation in $\mathbb{R}^4$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \tag{1}$$

with points $(x,t) = (x_1, x_2, x_3, t)$ in the domain

$$\Omega = \{(x,t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\},$$

bounded by the two characteristic cones

$$\Sigma_1 = \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\},$$

$$\Sigma_2 = \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2 + x_3^2} = t\}$$

and the ball

$$\Sigma_0 = \{t = 0, \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\},$$

centered at the origin $O : x = 0, t = 0$. The right-hand side function f of (1) satisfies some smoothness conditions in Ω, which will be fixed later. M. Protter [9] proposed the following BVPs:

Problem P1. *Find a solution of the wave equation* (1) *in* Ω *which satisfies the boundary conditions*

$$P1: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

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Problem P1*. *Find a solution of the wave equation* (1) *in* Ω *which satisfies the adjoint boundary conditions*

$$P1^*: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

Protter [9] formulated and studied the analogues of $P1$ and $P1^*$ in $\mathbb{R}^3$ as multidimensional analogues of the Darboux problem in the plane. It is known, that in contrast to the Darboux problems, the multidimensional Problem $P1$ is not well-posed, because its adjoint homogeneous Problem $P1^*$ has an infinite number of classical solutions ([4, 8, 11]). For recent known results concerning Protter's problems see papers [1, 6, 8] and references therein.

Let Y_n^m be the orthonormal system of *spherical functions* in $\mathbb{R}^3$. Expressed in Cartesian coordinates here, one can define them on the unit sphere $S^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ for $n \in \mathbb{N} \cup \{0\}$ by

$$Y_n^{2k}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Im}\left\{(x_2 + ix_1)^k\right\}, \quad \text{for} \quad k = 1, \dots, n;$$

$$Y_n^{2k+1}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Re}\left\{(x_2 + ix_1)^k\right\}, \quad \text{for} \quad k = 0, \dots, n,$$

where P_n are the *Legendre polynomials* and $C_{n,k}$ are constants such that functions Y_n^m form an orthonormal system in $L_2(S^2)$ (see [3]). For convenience in the discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation Y_n^m for the extended function, i.e. $Y_n^m(x) := Y_n^m(x/|x|)$ for $x \in \mathbb{R}^3 \backslash O$.

The known solutions of the the homogeneous Problem $P1^*$ are connected with the following functions: for $n, k \in \mathbb{N} \cup \{0\}$ define

$$H_k^n(x,t) = \sum_{i=0}^{k} A_{k,i}^n \frac{t(|x|^2 - t^2)^{n-1-k-i}}{|x|^{n-2i+1}}$$

where the coefficients are

$$A_{k,i}^n := (-1)^i \frac{(k-i+1)_i (n-k-i)_i}{i!(n-i+\frac{1}{2})_i}, \quad A_{k,0}^n = 1,$$

with $(a)_i := a(a+1)\cdots(a+i-1)$ and $(a)_0 := 1$.

Lemma 1 ([5]). *The functions*

$$V_{k,m}^n(x,t) = H_k^n(x,t) Y_n^m(x).$$

are classical solutions from $C^\infty(\Omega) \cap C(\overline{\Omega})$ *of the homogeneous Problem* $P1^*$ *for* $n \in \mathbb{N}$, $m = 1, \dots, 2n+1$ *and* $k = 0, 1, \dots, [(n-1)/2] - 2$.

From this result it follows that a necessary condition for the existence of classical solution for the Problem $P1$ is the orthogonality of the right-hand side function f to all functions $V_{k,m}^n(x,t)$. Alternatively, to avoid an infinite number of necessary conditions in the frame of classical solvability, in [6] we introduced *generalized solutions* for the Problem $P1$, eventually with a singularity at the origin O.

Definition 1 ([6]). *A function* $u = u(x,t)$ *is called a generalized solution of the Problem* $P1$ *in* Ω*, if the following conditions are satisfied:*

1) *$u \in C^1(\overline{\Omega}\backslash O)$, $u|_{\Sigma_0\backslash O} = 0, u|_{\Sigma_1} = 0$, and*

2) *the identity*

$$\int_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w) dx dt = 0$$

holds for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and in a neighborhood of Σ_2.

In general Problem $P1$ it is not classically solvable. In fact, it is known that for $n \in \mathbb{N}$ there exists a smooth right-hand side function $f \in C^n(\overline{\Omega})$, such that the corresponding unique generalized solution of Problem $P1$ has a strong power-type singularity at the origin O and behaves like $r^{-n}(P, O)$ there (see [8]). This feature deviates from the conventional belief that such BVPs are classically solvable for very smooth right-hand side functions f. It is interesting that the singularity is isolated only at a single point – the vertex O of the characteristic light cone, and does not propagate along the bicharacteristics which makes this case different from the traditional case of propagation of singularity (see for example Hörmander [2], Chapter 24.5).

Remark 1. *Notice that Lemma* 1 *shows that the Problem $P1$ is not Fredholm solvable. To explain the situation, let us define the operator*

$$T: \quad u_f \longmapsto f \in C^1(\overline{\Omega}),$$

where u_f is the unique classical solution to Protter Problem $P1$ for the right-hand side function f. Lemma 1 *gives* $\dim \operatorname{coker} T = \infty$ *and therefore T is not Fredholm operator in $C^k(\overline{\Omega})$ for example. However, one could expect it to be semi-Fredholm there (see [10]). A semi-Fredholm operator is a bounded operator that has a finite dimensional kernel* or *cokernel, and closed range. The results from [7] (see Theorem* 2*) show that the operator T is in fact a semi-Fredholm operator in $C^{10}(\overline{\Omega})$.*

Regarding to the exact behavior of the generalized solutions of Protter Problems, solutions with power-type singularity at O were found. From the other hand, some known estimates suggest that the solution may have at most exponential growth. However, it was not clear whether such solutions really exist.

In Section 2 we formulate recent results concerning the exact behavior of the generalized solution and semi-Fredholm solvability for the Protter Problem $P1$.

In Section 3 we present results from numerical experiments performed in search of generalized solution with exponential growth.

2. Asymptotic expansion and semi-Fredholm solvability. To study the exact behavior of the generalized solution, we started in [6] with the case when the right-hand side function $f \in C^1(\overline{\Omega})$ of equation (1) is a harmonic polynomial of order l with $l \in \mathbb{N} \cup \{0\}$, i.e.

$$f(x,t) = \sum_{n=0}^{l} \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x). \tag{2}$$

Denote by $\beta_{k,m}^n$ the parameters

$$\beta_{k,m}^n := \int_{\Omega} V_{k,m}^n(x,t) f(x,t) \, dx dt, \tag{3}$$

where $n = 1, \dots, l$; $k = 0, \dots, \left[\dfrac{n-1}{2}\right]$ and $m = 1, \dots, 2n+1$, then the main assertion in [6] is as follows.

Theorem 1 ([6])**.** *Suppose that the right-hand side function $f \in C^1(\overline{\Omega})$ has the form* (2)*. Then the unique generalized solution $u(x,t)$ of Problem $P1$ belongs to $C^2(\overline{\Omega}\backslash O)$ and has the following asymptotic expansion at the singular point $O : x = 0, t = 0$*

$$u(x,t) = \sum_{p=1}^{l} \left(|x|^2 + t^2\right)^{-p/2} F_p(x,t) + F(x,t),$$

where:
(i) *the function $F \in C^2(\overline{\Omega}\backslash O)$ and satisfies the a priori estimate*

$$|F(x,t)| \le C\,||f||_{C^1(\Omega)}\ , \quad (x,t) \in \Omega,$$

with constant C independent on f and $||f||_{C^k(\Omega)} = \sum\limits_{|\alpha|\le k} \max\limits_{\Omega} |D^\alpha f(x,t)|$;
(ii) *the functions F_p satisfy the equalities*

$$F_p(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{m=1}^{2p+4k+1} \beta_{k,m}^{p+2k} F_{k,m}^{p+2k}(x,t)\ , \quad p = 1, \dots, l,$$

with functions $F_{k,m}^n \in C^2(\overline{\Omega}\backslash O)$ bounded and independent on f;
(iii) *if at least one of the constants $\beta_{k,m}^{p+2k}$ in* (2) *is different from zero, then for the corresponding function $F_p(x,t)$ there exists a direction $(\alpha, 1) := (\alpha_1, \alpha_2, \alpha_3, 1)$ with $(\alpha,1)t \in \Sigma_2$ for $0 < t < 1/2$, such that*

$$\lim_{t\to+0} F_p(\alpha t, t) = c_p = \text{const} \ne 0.$$

If the right-hand side function f satisfies the orthogonality conditions $\beta_{k,m}^n = 0$ (see (3)), then from Theorem 1 it follows that the solution is bounded. Alternatively, according to the case (iii) each of the orthogonality condition (4) "controls" one power-type singularity.

In [7] we considered the more general case of smooth function $f(x,t)$ that no longer has the special form (2). We found some necessary and sufficient conditions for the existence of bounded solutions.

Theorem 2 ([7])**.** *Let the function $f(x,t)$ belong to $C^{10}(\overline{\Omega})$. Then, the necessary and sufficient conditions for existence of bounded generalized solution $u(x,t)$ of the Protter Problem $P1$ are*

$$\int_\Omega V_{k,m}^n(x,t) f(x,t)\, dxdt \;=\; 0, \tag{4}$$

for all $n \in \mathbb{N}$, $k = 0, \dots, \left[\dfrac{n-1}{2}\right]$, $m = 1, \dots, 2n+1$. The generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ and satisfies the a priori estimates

$$|u(x,t)| \le C\,\|f\|_{C^{10}(\overline{\Omega})}$$

and

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \le C(|x|^2 + t^2)^{-1} \, \|f\|_{C^{10}(\overline{\Omega})}$$

where the constant C is independent of the function $f(x,t)$.

3. Generalized solution with exponential growth. The generalized solutions have singularities at the point O if the orthogonality conditions (4) are not fulfilled. According to Theorem 1 only power type singularities are possible in the case when the right-hand side function is a harmonic polynomial. As for the general case, some previous results suggest that the solution may have at most exponential growth, but the question whether such solutions really exist was still open.

We performed numerical experiments in search of generalized solution u with exponential growth by suitably choice of the right-hand side function f. To achieve this, f breaches appropriately selected infinite subset of the orthogonal conditions (4). Using the asymptotic expansion formula from Theorem 1 and the properties of the special functions involved, the right-hand side function f is accordingly constructed so that $u(0,0,t,t)$ behaves like $e^{1/t}$ for small positive t.

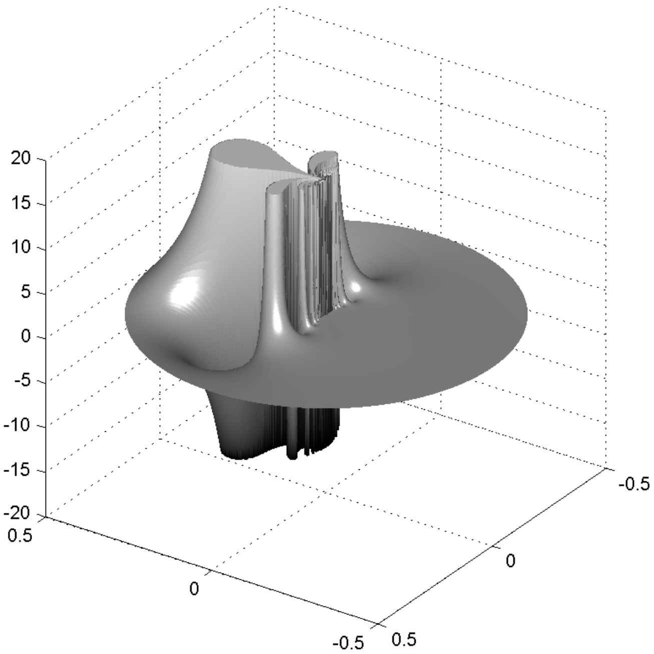

Fig. 1. Generalized solution with exponential growth

Figure 1 was created using MatLab and illustrates the behavior of the found generalized solution u of Protter Problem $P1$ with exponential growth. It represents the graphic of u on the part of the characteristic boundary cone $\Sigma_2 \cap \{x_1 = 0\}$. The singular point O is at the center and the outer age of the graphic corresponds to the circle

$S := \Sigma_1 \cap \Sigma_2 \cap \{x_1 = 0\}$. Notice that O also lies on the noncharacteristic part of the boundary – the ball Σ_0. Thus according to the homogenous boundary conditions any classical solution of the Problem $P1$ should vanish both on S and at the center O. On the other hand, although the constructed generalized solution u is zero on S, it has singularity at O. In fact, the computations show that the ridge on the left has exponential growth. Away from the point O, u is smooth – it belongs to $C^1(\overline{\Omega}\backslash O)$ and the singularity is isolated at O and does not propagate along the characteristic cone Σ_2. Let us mention some other interesting features of the graphic in the Figure 1. It looks like that u changes its sign infinitely many times near the point of singularity O, while on the other hand there are some directions on Σ_2 on which the solution is bounded at O.

As a result we formulate the following theorem.

Theorem 3. *There exists function $f \in C^1(\overline{\Omega})$ and a positive number $\delta \in (0, 1/2)$, such that the corresponding unique generalized solution $u(x_1, x_2, x_3, t) \in C^1(\overline{\Omega}\backslash O)$ of Problem $P1$ with right-hand function f satisfies the estimate*

$$u(0, 0, t, t) \geq \exp\left(\frac{1}{t}\right) \quad \textit{for} \quad 0 < t < \delta.$$

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Nedyu Popivanov, Todor Popov
Department of Mathematics and Informatics
University of Sofia
5, James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: nedyu@fmi.uni-sofia.bg, topopover@fmi.uni-sofia.bg

Rudolf Scherer
Institute for Applied and Numerical Mathematics
Karlsruhe Institute of Technology (KIT)
Englerstr. 2
76128 Karlsruhe, Germany
e-mail: rudolf.scherer@kit.edu

СИНГУЛЯРНИ РЕШЕНИЯ С ЕКСПОНЕНЦИАЛЕН РЪСТ ЗА ЧЕТИРИМЕРНОТО ВЪЛНОВО УРАВНЕНИЕ

Недю И. Попиванов, Тодор П. Попов, Рудолф Шерер

Разглеждат се четиримерни гранични задачи за нехомогенното вълново уравнение. Те са предложени от М. Протер като многомерни аналози на задачата на Дарбу в равнината. Известно е, че единственото обобщено решение може да има силна степенна особеност само в една гранична точка. Тази сингулярност е изолирана във върха на характеристичния конус и не се разпространява по конуса. Друг аспект на проблема е, че задачата не е фредхолмова, тъй като има безкрайномерно коядро. Предишни резултати сочат, че решението може да има най-много експоненциален ръст, но оставят открит въпроса дали наистина съществуват такива решения. Показваме, че отговора на този въпрос е положителен и строим обобщено решение на задачата на Протер с експоноциална особеност.