*МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2011*
*MATHEMATICS AND EDUCATION IN MATHEMATICS, 2011*
*Proceedings of the Fortieth Jubilee Spring Conference*
*of the Union of Bulgarian Mathematicians*
*Borovetz, April 5–9, 2011*

# COMPARATIVE ANALYSIS OF THE OPPOSITE ROTATING CYLINDERS IMPACT OVER COUETTE FLOW FOR RAREFIED GAS*

**Peter Gospodinov, Dobri Dankov, Vladimir Roussinov, Stefan Stefanov**

The cylindrical Couette flow of a rarefied gas is studied in the case of both rotating cylinders with equal velocities but with opposite direction. The purpose of this study is to determine the influence of small speeds on macro-characteristics – $\rho$, $V$, $T$. The numerical results are obtained using the Direct Simulation Monte Carlo (DSMC) method and numerical solution of Navier-Stokes equations for relatively small (subsonic) speeds. The results obtained by both methods are in an excellent agreement for a small Knudsen number Kn = 0.02. It was found that there was "fixed" point for density and velocity. These results are important for applications in non-planar microfluidic problems.

**1. Introduction.** Fluid transport in micro and macro channels yields the necessity to study flow in a cylindrical coordinate system. Note that Couette cylindrical flow is a fundamental problem in the rarefied gas dynamics [1, 6, 9, 10, 11, 12, 13, 15]. As such, its modeling and numerical solving is of a great importance for the microfluidics, which is the theoretical background for analysis of new emerging Micro Electro Mechanical Systems MEMS [2, 3, 14].

The design of adequate mathematical models of gaseous flows in micro devices is one of the most important tasks of the studies. We consider both molecular and continuum models treating the gaseous flow by using different level of mathematical description. Both models take into account the specific microfluidic effects of gas rarefaction and slip-velocity regime at the solid boundaries.

In the present paper we compare results obtained by using the molecular Direct Simulation Monte Carlo (DSMC) method with those produced by a numerical solution of the continuum Navier-Stokes equations for compressible flow (NS) [7, 8].

Both methods are used to model the cylindrical Couette flow for Knudsen number 0.02, 0.06, 0.1. Cylinder velocities are relative small (subsonic) with the same value but with opposite directions. The aim of the present paper is to study the gas conduction in the gape depending on the variety subsonic velocity boundary conditions and to establish the field of matching decisions in the two methods in terms of number of Knudsen and cylinder velocities.

---

*2000 Mathematics Subject Classification: 65C20.
**Key words:** Fluid Dynamics, Kinetic theory, Rarefied gas, DSMC.
This work is supported by the NSF of Bulgaria under Grant No DID 02/20 – 2009.

**2. Formulation of the problem and methods of solution.** We study a rarefied gas flow between two coaxial cylinders (one dimensional, axis-symmetrical problem) with equal temperatures $T_1 = T_2$. The inner cylinder has radius $R_1$ and the outer – $R_2$. The outer cylinder rotates with a constant velocity $V_2$ and the inner one – with constant velocity $V_1$. Figure 1 shows a three-dimensional version, but the real studies are one-dimensional along axis $r$.

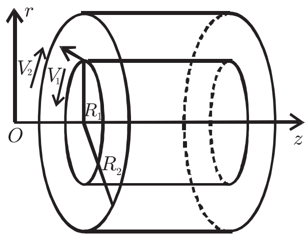



Fig. 1. Flow geometry

**3. Continuous Model and Numerical Simulation.** The continuous model is based on the Navier-Stokes equations for compressible fluid, completed with the equations of continuity and energy transport. The governing equations are written as follows:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = -\frac{\partial P}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{\tau_{\varphi\varphi}}{r} \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\varphi}\right) \tag{3}$$

$$\rho c_P \frac{DT}{Dt} = \operatorname{div}(\lambda \operatorname{grad} T) - P \operatorname{div}\mathbf{V} + \mu\Phi \tag{4}$$

$$P = \rho R T, \tag{5}$$

where $\mathbf{V}$ is the velocity vector, $u$ and $v$ are the velocity components along axis $r$ and $\varphi$. A rather standard notation is used in Eqs (1)–(5): $\rho$ is density and $T$ is the temperature. $\rho$, $P$, $T$, $u$, $v = f(r,t)$. $\tau_{i,j}$ are the stress tensor components and $\Phi$ is the dissipation function [16]. For a perfect monatomic gas, the viscosity and the coefficient heat transfer read as [13]:

$$\mu = \mu(T) = C_\mu \rho_0 l_0 V_0 \sqrt{T}, \quad C_\mu = \frac{5}{16}\sqrt{\pi} \tag{6}$$

$$\lambda = \lambda(T) = C_\lambda \rho_0 l_0 V_0 \sqrt{T}, \quad C_\lambda = \frac{15}{32}\sqrt{\pi} \tag{7}$$

The above written equations are normalized by using the following scales: for density, $\rho_0 = mn_0$ ($m$ – is the molecular mass, $n_0$ – the average number density), for velocity $V_0 = \sqrt{2RT_0}$ – $R$ is the gas constant, for length – the distance between the cylinders $L = R_2 - R_1$, for time $t_0 = L/V_0$, for temperature $T_0 = T_w$ – the wall temperature of both cylinders. The Knudsen number is $\mathrm{Kn} = l_0\mathrm{L}$, where the mean free path is $l_o$ and $\gamma = c_P/c_V = 5/3$ ($c_P$ and $c_V$. are the heat capacities at constant pressure and constant volume respectively). In this way in the dimensionless model the characteristic number Kn and the constants $C_\mu$ and $C_\lambda$ take part. After the scaling, the same symbols for the dimensionless $r$, $t$, $\rho$, $P$, $T$, $u$, $v$ and $R_i$ are used.

For the problem (1)–(4), first-order slip boundary conditions are imposed at both walls, which can be written directly in dimensionless form as follows [14, 15]:

$$v \mp 1.1466\mathrm{Kn}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = \bar{v}_i, \tag{8}$$

$$u = 0, \tag{9}$$

$$T \pm 2.1904\mathrm{Kn}\frac{\partial T}{\partial r} = \bar{T}_i, \tag{10}$$

at $r = R_i$, $i$=1, 2. In Eqs. (8)–(10) $\bar{v}_i = V_i/V_0$ and $\bar{T}_i = T_{W,i}/T_0 = 1$ are the dimensionless wall velocity and temperature for both cylinders. The equations of transfer (1)–(4), together with the boundary conditions (8)–(10), and zero initial distributions for $u, v$ and $T$, formulate the initial unsteady state boundary-value problem. A second order of approximation, implicit finite difference scheme to solve numerically the formulated problem is used [17]. Thus the difference value problem, for a given time $t$ is reduced to the solution of 4 linearized systems of M algebraic equations. The obtained algebraic system has a diagonal and weakly filled matrix. Due to the problem non-linearity, additionally an internal iteration process is used.

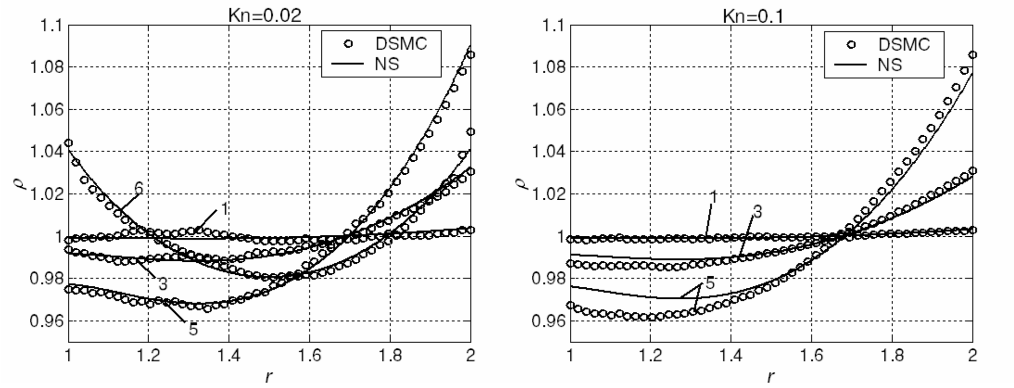
**4. Direct Simulation Monte Carlo (DSMC) Method.** The gas considered is simulated as a stochastic system of $N$ particles [4, 5]. All quantities used are non-dimensional, so that the mean free path at equilibrium is equal to 1. The basic steps of simulation are as follows:

A. The time interval $\left[0;\hat{t}\right]$ over which the solution is found, is subdivided into subintervals with step $\Delta t$.
B. The space domain is subdivided into cells with sides $\Delta z$, $\Delta r$. For one-dimensional problem along the axis z is one cell, (z is the axial direction on Fig. 1).
C. Gas molecules are simulated in gap G using a stochastic system of $N$ points (particles) having position $z_i\left(t\right)$, $r_i\left(t\right)$ and velocities $\overrightarrow{\xi_i\left(t\right)}$
D. $N_m$ particles are located in the $m$-th cell at any given time. This number varies during the computer simulation by the following two stages:
   Stage 1. Binary collisions in each cell are calculated, whereas particles do not move. Collision modeling is realized using Bird's scheme "no time counter".
   Stage 2. Particles move with new initial velocities acquired after collisions, and no external forces act on particles. No collisions are accounted for at this stage.
E. Stage 1 and Stage 2 are repeated until $t = \hat{t}$.
F. Flow macro-characteristics (density, velocity, temperature) are calculated as time-averaged when steady regime is attained.
G. Boundary conditions are diffusive over the cylinders and periodical along axis $Oy$.

All magnitudes used are non-dimensional so that the mean free path in equilibrium state is equal to 1. The modeling particles number for DSMC method is 4000000.

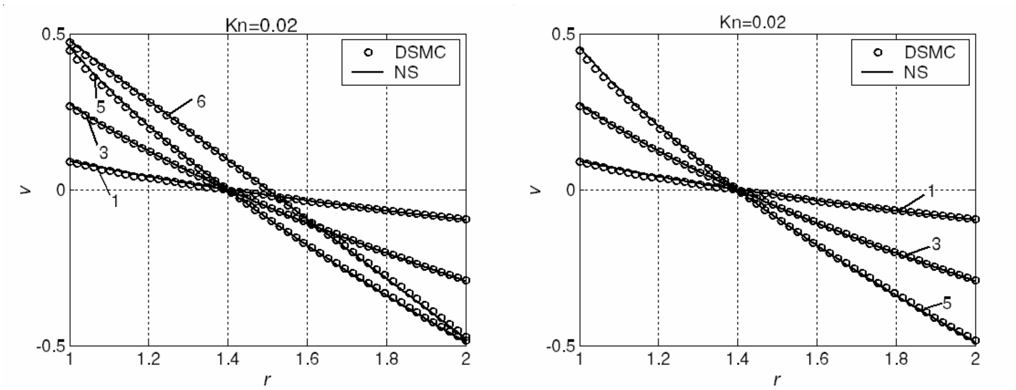
**5. Numerical results.** We study the fife typical cases of rarefied gas between rotating cylinder:

Case 1: $V_1 = 0.1$, $V_2 = -0.1$, $T_1 = T_2 = 1$, $R_1 = 1$, $R_2 = 2$;



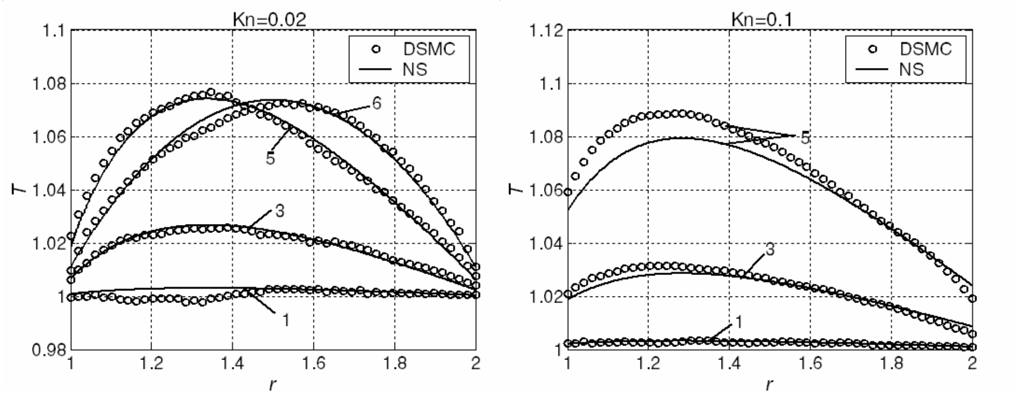

(A) (B)

Fig. 2. Density profile in the cases 1, 3, 5 and




(A) (B)

Fig. 3. $\varphi$ velocity profile in the cases 1, 3, 5 and 6




(A) (B)

Fig. 4. Temperature profile in the cases 1, 3, 5 and 6

Case 2: $V_1 = 0.2$, $V_2 = -0.2$, $T_1 = T_2 = 1$, $R_1 = 1$, $R_2 = 2$;
Case 3: $V_1 = 0.3$, $V_2 = -0.3$, $T_1 = T_2 = 1$, $R_1 = 1$, $R_2 = 2$;
Case 4: $V_1 = 0.4$, $V_2 = -0.4$, $T_1 = T_2 = 1$, $R_1 = 1$, $R_2 = 2$;
Case 5: $V_1 = 0.5$, $V_2 = -0.5$, $T_1 = T_2 = 1$, $R_1 = 1$, $R_2 = 2$;
Case 6: $V_1 = 0.2$, $V_2 = -0.4$, $T_1 = T_2 = 1$, $R_1 = 1000$, $R_2 = 1001$
for Kn=0.02, 0.06, 0.01.

The results obtained by both methods are: in an excellent agreement at a small Knudsen number Kn = 0.02 – Figure 2A, 3A and 4A and Kn = 0.06; in a satisfactory agreement at 0.1 – Figure 2B, 3B and 4B. The flow character is maintained at increasing the Knudsen number while the differences are in the macro-characteristics value. Very good matching results are calculated for velocities less than 0.3 and for all studied values of Knudsen number.

Planar case is studied to determine the curvature influence on the flow macro-characteristics – Figure 2A, 3A and 4A. Planar case is studied only for Kn = 0.02 because there the two methods show very good match.

There is a "fixed" point for density and velocity in all cases investigated in a fixed number of Knudsen – Figure 2 and 3. "Stationary" point for the density is 1, and the velocity is 0. This value for the studied cases have the same coordinate $r$. Typical of this point is that with increasing number of Kn, $r$ coordinate decreases (closer to the inner cylinder). This fact can be used in MEMS designing.

## REFERENCES

[1] C. Cercignani, F. Sernagiotto. Cylindrical Couette Flow of a Rarefied Gas *Physics of Fluids*, **10** (1967), 1200.

[2] K. W. Tibbs, F. Baras, A. L. Garcia. Anomalous flow profile due to the curvature effect on slip length= *Phys, Rev. E*, **56** (1997), 2282–2283.

[3] K. Aoki, H. Yoshida, T. Nakanishi, A. Garcia. , Inverted velocity profile in the cylindrical Couette flow of a rarefied gas. *Phys. Rev. E*, **68** (2003), 016302.

[4] G. A. Bird. Molecular Gas Dynamics and the Direct Simulation of Gas Flows. Oxford University Press, Oxford, 1994.

[5] C. Cercignani. Rarefied Gas Dynamics, From Basic Concepts to Actual Calculations. Cambridge University Press, Cambridge, 2000.

[6] S. K. Stefanov, V. M. Roussinov, C. Cercignani. Rayleigh-Benard flow of a rarefied gas and its attractors. Part 1: Convection regime. *Physics of Fluids*, **14** (2002), 2255–2269.

[7] S. K. Stefanov, V. M. Roussinov, C. Cercignani. Rayleigh-Benard flow of a rarefied gas and its attractors. Part 2: Chaotic and periodic convective regimes. *Physics of Fluids* **14**, 2270-2288 (2002)

[8] D. Y. Dankov, V. M. Roussinov. Modeling and applications of the cylindrical Couette flow of a rarefied gas. *AIP Conf. Proc.*, **1067** (2008), 239.

[9] P. Gospodinov, D. Dankov, V. Roussinov, S. Stefanov. Modeling of Cylindrical Couette Flow of Rarefied Gas. The Case of Rotating Outer Cylinder 1AMiTaNS'09. AIP Conf. Proc., **1186** (2009), 200–206

[10] P. Gospodinov, D. Dankov, V. Roussinov, S. Stefanov. Modeling of cylindrival Couette flow of rarefied gas between rotating cylinders. *Math. and Education in Math.* **39** (2010), 184–189.

[11] G. P. Neitzel. Marginal stability of impulsively initiated Couette and spin decay. *Journal of Fluid Mechanics*, **123** (1982).

[12] G. H. Tang, X. J. Gu, R. W. Barber, D. R. Emerson, Y. H. Zhang. Lattice Boltzmann simulation of nonequilibrium effects in oscillatory gas flow. *Phys. Rev. E*, **78** (2008), 026706.

[13] M. Malik, J. Dey, Meheboob Alam. Linear stability, transient energy growth and the role of viscosity stratification in compressible plane Couette flow. *Phys. Rev. E*, **77** (2008), 036322.

[14] Y. Hiroaki, A. Kazuo. Linear stability of the cylindrical Couette flow of a rarefied gas. *Phys. Rev. E*, **73** (2006), 021201.

[15] S. Stefanov, P. Gospodinov, C. Cercignani. Monte Carlo Simulation and Navier-Stokes Finite Differnce Solution of Rarefied Gas Flow Problems. *Physics of Fluids*, **10** (1998), Issue 1, 289–300.

[16] A. Manela, I. Frankel. On the compressible Taylor-Couette problem. *J. Fluid Mech.*, **588** (2007), 59–74.

Peter N. Gospodinov, Dobri Y. Dankov
Vladimir M. Roussinov, Stefan K. Stefanov
Institute of Mechanics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 4
1113 Sofia, Bulgaria
e-mail: png@imbm.bas.bg, dankov@imbm.bas.bg
vladimir@imbm.bas.bg, stefanov@imbm.bas.bg

## СРАВНИТЕЛЕН АНАЛИЗ НА ВЛИЯНИЕТО НА СРЕЩУПОЛОЖНО ВЪРТЯЩИ СЕ ЦИЛИНДРИ ЗА ТЕЧЕНИЕ НА КУЕТ ЗА РАЗРЕДЕН ГАЗ

**Петър Господинов, Добри Данков, Владимир Русинов, Стефан Стефанов**

Изследвано е цилиндрично течение на Кует на разреден газ в случая на въртене на два коаксиални цилиндъра с еднакви по големина скорости, но в различни посоки. Целта на изследването е да се установи влиянието на малки скорости на въртене върху макрохарактеристиките – $\rho$, $V$, . Числените резултати са получени чрез използване на DSMC и числено решение на уравненията на Навие-Стокс за относително малки (дозвукови) скорости на въртене. Установено е добро съвпадение на резултатите получени по двата метода за Kn = 0.02. Установено е, че съществува "стационарна" точка за плътността и скоростта. Получените резултати са важни при решаването на неравнини, задачи от микрофлуидиката с отчитане на ефектите на кривината.

**Ключови думи:** Механика на флуидите, Кинетична теория, Разреден газ, DSMC