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ANALYTIC METHODS – USEFUL TOOL FOR OBTAINING LONG-LASTING STRUCTURAL KNOWLEDGE*

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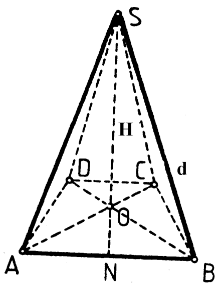
This work attempts to introduce analytical methods into the mathematics curriculum. Analytic methods can stimulate the process of creative thinking and motivate students in their current learning. All this should contribute, above all, to the possibility for students to obtain long-lasting structural knowledge. But, do teachers really pay attention to structural knowledge in their everyday teaching? In this paper it is included a questionnaire with qualitative answers analysis. The questionnaire facilitated the detection of the situation giving a clear signal that in our conditions the efforts for long-lasting structural knowledge are more declarative than realistic.

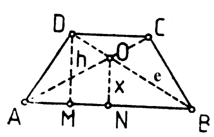
With the analytical method for solving tasks we start from what is requested in the task. The basic question the student needs to ask is:

"What do I need to know in order to fulfill the requirement of the task?"

We can answer this question correctly only if we properly understand the condition of the task and if we recognize the relations which connect the condition with the posed question?



Picture 1



Picture 2

Example 1. Calculate the volume of a pyramid whose base is an isosceles trapezoid with bases $a = 5$, $b = 3$ and leg $c = 7$, if the height of the pyramid goes through the intersect point of the base's diagonals and the bigger slant edge is $d = 10$.

Solution. 1) We calculate the volume of the pyramid according to the formula $V = \frac{BH}{3}$, where B is the area of the pyramid's base $ABCD$, and H – its height which we need to calculate (Picture 1).

2) We calculate the area of the pyramid's base according to the formula for calculating area of isosceles trapezoid $B = \frac{a+b}{2}h$, where h is the height of the trapezoid (Picture 2).

3) In order to calculate the height h of the isosceles trapezoid we use the Pythagorean theorem: $h = \sqrt{c^2 - \left(\frac{a-b}{2}\right)^2} = \sqrt{7^2 - 1^2} = \sqrt{48} = 4\sqrt{3}$.

*Key words: analytic methods, structural and long-lasting knowledge, reflaction.

4) In order to calculate the height of the pyramid we use again the Pythagorean theorem. We have $d = \overline{BD}$ and if we denote $e = \overline{OB}$ (Picture 2), then we get $H = \sqrt{d^2 - e^2}$. According to this, we have to calculate $e = \overline{OB}$.

5) In order to calculate $e = \overline{OB}$, we use once again the Pythagorean theorem for $\Delta OBN$and we get $e = \sqrt{x^2 + \left(\frac{a}{2}\right)^2}$. Further, we have to determine $x = \overline{ON}$ (Picture 2).

6) From $\Delta BMD \sim \Delta BON$ we calculate $x : h = \frac{a}{2} : \left(\frac{a-b}{2} + b\right)$ and if we consider the result from step 3), then we get:

$$x = \frac{ah}{a+b} = \frac{5 \cdot 4\sqrt{3}}{5+3} = \frac{5\sqrt{3}}{2}.$$

7) Now, if we consider results 3) and 6), returning to 5), 4), 2) and 1), then we consequently get:

$$e = \sqrt{x^2 + \left(\frac{a}{2}\right)^2} = \sqrt{\left(\frac{5\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = 5, \quad H = \sqrt{d^2 - e^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3},$$

$$B = \frac{a+b}{2}h = \frac{5+3}{2} \cdot 4\sqrt{3} = 16\sqrt{3} \text{ and } V = \frac{BH}{3} = \frac{16\sqrt{3} \cdot 5\sqrt{3}}{3} = 80. \qquad \Box$$

Example 2. Several passengers got in a bus at the first bus stop. At the second bus stop, half of them got off, and two people got on. At the third bus stop, half of the passengers got of, and three people got on. At the fourth bus stop, half of the passengers got off, and only four people got on. And at that moment there were 7 passengers in total. How many passengers got on the bus at the first bus stop?

Solution. If this task is assigned to students from higher grades from primary education, they will solve it using the method of algebraic analysis discussed in the next point.

We denote the number of passengers who got on the bus at the first bus stop by x. Subsequently, after the second bus stop we have $x : 2 + 2$, and after the third bus stop passengers $(x : 2 + 2) : 2 + 3$. According to this, after the fourth bus stop we have $[(x : 2 + 2) : 2 + 3] : 2 + 4$ passengers, and thus we get the following equation

$$[(x : 2 + 2) : 2 + 3] : 2 + 4 = 7,$$

and the solution is $x = 8$.

However, we can solve this task by using the analytical method and thus this type of solving a task is acceptable even for students from 4^{th} grade. Namely, we start with the number of passengers after the fourth bus stop and we move backwards, and consequently we calculate the number of passengers after the third, second and first bus stop.

Four passengers got on the bus at the fourth bus stop and because there were 7 afterwards, we conclude that before these passengers got on there were 3 passengers; that is the half of the number of passengers who travelled between the third and fourth bus stop which means that this number equals 6. Accordingly, we have $7 \to 3 \to 6$.

At the third bus stop, half of the passengers got off and three got on and since there were 6 passengers, we get $6 \to 3 \to 6$. Consequently, before the third bus stop the number of passengers was 6.

At the second bus stop, half of the passengers got off and two got on, consequently we get $6 \to 4 \to 8$. Finally, the number of passengers that travelled between the first and the second bus stop is 8, which means that 8 passengers got on the bus at the first bus stop.

The previously stated explanation can be written down using the following format

$$7 \to 3 \to 6 \to 3 \to 6 \to 4 \to 8. \qquad \square$$

As we can see, the representation of the solution with the analytical method is not as concise as the synthetic method. Nevertheless, we can say that the analytical method has a range of advantages such as:

– Students independently discover the solution of the task, process which contributes to the development of mathematical thinking with students,
– Discovering various pathways for solving one and the same task, etc.

The analytical method has several variations, two of which, we would like to mention here are method of algebraic analysis and method of analysis when solving constructive tasks in geometry.

Method of algebraic analysis. This method comprises of all the methods for solving algebraic tasks. The most important one is the method of equations and, therefore, usually when we talk about algebraic method we specifically have in mind this method. The method of equations can be divided into three stages.

In the **first stage** we introduce one or more variables and based on the condition and request of the task we construct an equation or system of equation (accessory task). According to this, in the first stage, the basic task is transformed into an accessory task.

The **second stage** of this method consists of series of transformations of the accessory task, with the help of which we get the roots of the primary task, i.e. solutions to the primary system.

In the **third stage** we verify whether the requested corresponds to the obtained set of solutions. If these sets do not correspond, then we look for the conditions which need to be fulfilled during the series of transformations of the initial equation or system of equations in order to get the set of solutions to the basic task. In other words, we need to verify whether each of the obtained equations is equivalent to the initial.

The basic equation is assigned and a unconstructed equation or system of equations. If we follow the principle of equivalence in the series of transformations, then the process of solving the accessory task may be written in the following form

$$(1) \qquad A(X) \Leftrightarrow A_1(X) \Leftrightarrow A_2(X) \Leftrightarrow \cdots \Leftrightarrow A_{k-1}(X) \Leftrightarrow A_k(X)$$

and in this case the requested corresponds to the obtained solution.

However, in practice the newly-obtained equations are most often a consequence of the initial. Namely, if in form (1) of solving the accessory task the equivalence $A_r \Leftrightarrow A_{r+1}$ is substituted with the implication $A_r \Rightarrow A_{r+1}$, then the last equation contains all the solutions of the basic equation, but it can also contain solutions which are not solutions of the initial equation, or of the initial task as well. Therefore, we need to verify every solution with the conditions from the initial task. In practice, the teacher's insistence that students understand the necessity of making this verification and performing the same is of great importance.

Example 3. While running away from a dog, a fox made 60 jumps before the dog started chasing it. The fox makes 3 jumps, while the dog makes 2 jumps. Apart from this, we also know that the length of 7 jumps made by the fox is the same as 3 dog's jumps. How many jumps must the dog make in order to catch the fox?

Solution. ***First stage.*** The dog makes x jumps until it reaches the fox. During this time the fox makes $\frac{3}{2}x$ jumps, which together with the advantage it had in the beginning is $\frac{3}{2}x + 60$ jumps. We know that the length of 7 jumps made by the fox is the same as 3 dog's jumps, therefore,

$$\frac{\frac{3}{2}x + 60}{7} = \frac{x}{3}. \tag{2}$$

Second stage. From equation (2) with equivalent transformation we consequently get:

$$\frac{3x+120}{14} = \frac{x}{3} \quad \Leftrightarrow \quad 3(3x+120) = 14x \quad \Leftrightarrow \quad 9x + 360 = 14x \quad \Leftrightarrow \quad 5x = 360 \quad \Leftrightarrow \quad x = 72.$$

Third stage. If the dog makes 72 jumps, then their length is the same as $(72 : 3) \cdot 7 = 24 \cdot 7 = 168$ fox's jumps. For that time the fox will make $(72 : 2) \cdot 3 = 36 \cdot 3 = 108$ jumps, and because at the beginning it had an advantage of 60 jumps, we get that the fox will be at a distance of $108 + 60 = 168$ own jumps and the place where the dog was positioned at the beginning. Therefore, the dog must make 72 jumps in order to catch up on the fox. □

Example 4. Calculate all the values for parameter k, for which the equation $\lg(x^2 + 2kx) - \lg(8x - 6k - 3) = 0$ has multiple square root, and afterward calculate the root.

Solution. ***First stage.*** We transform the given equation into a quadratic equation

$$x^2 + 2(k-4)x + 6k + 3 = 0. \tag{3}$$

The last equation has a double root if and only if its discriminant is equal to zero i.e. if and only if

$$4(k-4)^2 - 4(6k+3) = 0. \tag{4}$$

Second stage. The equation (4) is equivalent to the following equation

$$k^2 - 14k + 13 = 0$$

The solutions are $k_1 = 1$ and $k_2 = 13$.

For $k_1 = 1$, from (3) we solve the equation $x^2 - 6x + 9 = 0$, and get $x_1 = x_2 = 3$.

For $k_2 = 13$, from (3) we calculate the equation $x^2 + 18x + 81 = 0$, and we get $x_1 = x_2 = -9$.

Third stage. If, in the initial equation, we replace $k = 1$ and $x = 3$, then we get $\lg(3^2 + 2 \cdot 1 \cdot 3) - \lg(8 \cdot 3 - 6 \cdot 1 - 3) = \lg 15 - \lg 15 = 0$, which means that one of the solutions to the equation is $k = 1$ and $x = 3$.

For $k = 13$ and $x = -9$, the expressions under the second logarithm symbol in the initial task gets the value of -153. For this value the logarithm is not defined, which means that we have a solution to the equation (3) which is not a solution of the initial equation and this results from the fact that

$$\lg(x^2 + 2kx) - \lg(8x - 6k - 3) = 0 \Rightarrow \quad x^2 + 2(k-4)x + 6k + 3 = 0,$$

but the reverse implication is false. Finally, the only solution to the starting task is $k = 1$ and $x = 3$. □

Method of analysis when solving constructive tasks. As we previously stated, one of the variations of the analytical method occurs when solving constructive tasks. That is to say, if one constructive task is complex and the pathway to its solution is not obvious, then we perform the following procedure in order to solve it which comprises of 4 stages: analysis, construction, proof and discussion.

Analysis is the first stage in solving constructive tasks and it comprises of drawing the requested shape as if we are familiar with it. The aim of this stage is to discover the connection between the given and requested elements of the shape we need to draw. Simultaneously, we discover the procedure or method to solve the task in the most appropriate manner. Generally, the analysis starts by sketching a drawing and almost always with the words "Let us assume that the task is solved". After that, we detect the known elements on the sketch, separate simpler shapes (or form them by making additions on the sketch) and look for a supporting shape which will satisfy the following two conditions:

1) the shape can be constructed from the given elements of the basic task and

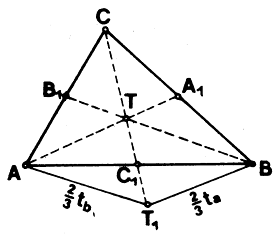
2) starting from it, we can construct the requested task.

Construction is the second stage carried out on the basis of the performed analysis and, frequently, it directly results from the analysis.

Proof is the third stage in this method and it aim is to prove that the shape we have constructed satisfies the conditions of the task.

Discussion is the fourth stage in this method, and here we verify in what way the solution of the task depends from the given elements and whether this is unique or not, i.e. with the given elements, can we construct only one or more noncongruent shapes, which are solutions to the task.

Example 5. Construct a triangle with the given medians t_a, t_b and t_c.



Picture 3

Solution. ***Analysis.*** The requested triangle ABC with medians t_a, t_b and t_c is constructed and T is the triangle's centroid (Picture 3).

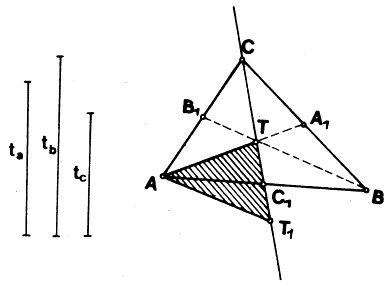
It is known that the centroid T cuts every median in the ratio 2 : 1 (starting from the vertex). We continue the median t_c through point C_1 for $\frac{1}{3}t_c$. If we connect the obtained point T_1 with vertices A and B, then we get a parallelogram AT_1BT, where the sides are $\frac{2}{3}t_a$ and $\frac{2}{3}t_b$, and the diagonals are $\frac{2}{3}t_c$ and $\overline{AB} = c$.

Construction. We construct ΔAT_1T with line segments $\frac{2}{3}t_a$, $\frac{2}{3}t_b$ and $\frac{2}{3}t_c$, we draw a median AC_1, which is half of the side AB, and we use this condition to calculate the vertex B. We will get vertex C if side TT_1 is continued through vertex T for $\overline{TT_1} = \frac{2}{3}t_c$ length.

Proof. It is sufficient to prove that the obtained triangle has medians t_a, t_b and t_c. Since $\overline{TT_1} = \frac{2}{3}t_c$ and C_1 is the middle of TT_1, we get that point T divides the median to ΔABC in $2:1$ ratio. It follows from

$$\overline{BT} = \overline{AT_1} = \frac{2}{3}t_b i \overline{AT} = \overline{BT_1} = \frac{2}{3}t_a.$$

That ΔABC has medians t_a, t_b and t_c.



Discussion. If we can construct a triangle with medians t_a, t_b and t_c, then ΔAT_1T is uniquely determined, thus the task has only one solution. □

Do teachers pay attention to structural knowledge in their everyday teaching? For the needs of this survey a questionnaire including the discussed tasks was prepared. I would like to analyze the obtained answers from the questionnaire. The questionnaire was distributed to more than thirty successful teachers from primary and secondary schools in Macedonia, and in continuation I describe the conclusions reached from their answers.

The questionnaire facilitated the detection of the situation giving a clear signal that in our conditions the efforts for long-lasting and structural knowledge are more declarative than realistic. To support this I will mention the most important conclusions:

Although everyone generally knows that the class consists of three parts and everyone makes an effort to implement the same, as well as the usage of reflection, only a few of the teachers provides a successful inspection of homework.

During the introduction of new material, as well as assessment of newly learnt material in class, teachers use short, simple questions which neither incite nor detect reflection and are in function of fulfilling the teacher's aims rather than assisting the learning process and memorization of the material.

Teachers focus on receiving correct answers rather than explaining the mistakes made by the students.

Answering on the basis of expected outcome, and not realistic solving a task, namely, instead of concentrating on solving the task, it is assumed that the answer to the task's question whether the solution will be correct is negative, which is completely wrong. The successful solving must be "exempt from" the expected outcome because the procedure provides the requested answer.

Skipping steps from the process of solving mathematical tasks in the interest of saving time is methodologically incorrect.

It is unproductive to provide full solutions without indicating an example for checking the previously explained procedure because by watching or reading students memorize only a small part from the essence for obtaining structural knowledge.

Almost all old teachers, included in the survey, give tests which last entire school hour, instead of 20 minute short assessments, ignoring the fact of the extent and manner in which "the deviations from the path of cognition are significant for the students" , as well as for the process of acquiring cognitive and permanent knowledge.

Conclusion. The mentioned examples show that the teacher's excellent beforehand preparation, i.e. correctly introduced and chosen tasks are an important but not a sufficient condition for a successful lesson. The teacher must persistently stimulate students, check their individual work, and ask for answers connected to questions like: Why? Does this function or not? Solve the following task according to the shown example! What's your conclusion? ... One important element in obtaining long-lasting knowledge is stimulating a discussion connected to the steps from the solution and openly discussing the same. What's more, we should avoid formal questions, questions which can be answered with a simple yes or no, and most importantly, we should stimulate explanations for the choice of steps or the solution.

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АНАЛИТИЧНИТЕ МЕТОДИ КАТО СРЕДСТВО ЗА ПРИДОБИВАНЕ НА ТРАЙНИ СТРУКТУРНИ ЗНАНИЯ

Валентина Гоговска

В този труд се разглежда въвеждането на аналитичния метод като средство за получаване на трайни структурни знания. Дали в ежедневната си работа учителите по математика наистина разчитат на структурни знания? За да получим отговора на този въпрос, проведохме анкета с подходящо направен въпросник. Анкетата е проведена с учители в основното и средното училище в Република Македония. Анализът на учителските отговори дава ясен сигнал, че усилията за получаване на трайни структурни знания са предимно на ниво обсъждане, а не приложими в реалността.