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DEMONSTRATING THE BIRTH OF AN IDEA TO A HIGH SCHOOL STUDENT

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We propose one way to introduce high school students to how a general mathematical statement can be obtained *via* observations of several partial cases of given problems in elementary number theory.

Let us consider some typical problems that often appear in various mathematical contests:

Problem 1. What is the last digit of 2^{2011} (where 2011 is usually replaced by the current year when the contest is being held)?

Problem 2. For what natural power n of 2, 8 appears as a last digit for the 33rd time?

Problem 3. What is the last digit of 372^{45}?

There are more or less "standard" solutions using modular arithmetic. Here we try to show how one can demonstrate the birth of these ideas in the classroom thus making students to witness the way of the mathematical discovery and teaching them to find the general rule after observing partial cases.

The most natural approach is to write down the consequent increasing natural powers of 2 and we do this grouping them in four on Table 1.

We now try to observe some interrelations between the power y of 2 and the last digit in the decimal representation of 2^y. It is easy to see that the last digit seems to appear periodically in each group in the order – 2, 4, 6, 8.

Let us write down the powers y in increasing order, grouping them accordingly to the last digit in the decimal representation of 2^y:

(2) → 1, 5, 9, 13, 17, ...

(4) → 2, 6, 10, 14, 18, ...

(8) → 3, 7, 11, 15, 19, ...

(6) → 4, 8, 12, 16, 20, ...

We notice that in each group the difference between two consecutive members is 4, i.e. each group is an arithmetic progression with a difference 4 and different first member. This leads to the following natural question:

Q1: If 2^n ends on 2, 4, 8 or 6 correspondingly, then what will be the last digit of 2^{n+4}?

Table 1

2^y	y	2^y	No of the group of four
2^1	**1**	**2**	1
2^2	***2***	***4***	1
2^3	**3**	**8**	1
2^4	***4***	1***6***	1
2^5	**5**	3**2**	2
2^6	***6***	6***4***	2
2^7	**7**	12**8**	2
2^8	***8***	25***6***	2
2^9	**9**	51**2**	3
2^{10}	***10***	102***4***	3
2^{11}	**11**	204**8**	3
2^{12}	***12***	409***6***	3
2^{13}	**13**	819**2**	4
2^{14}	***14***	1638***4***	4
2^{15}	**15**	3276**8**	4
2^{16}	***16***	6753***6***	4
2^{17}	**17**	13507**2**	5
2^{18}	***18***	27014***4***	5
2^{19}	**19**	54028**8**	5
2^{20}	***20***	108057***6***	5

Since $2^{n+4} = 2^n.2^4 = 2^n.16$ then if $(\ldots 2)$, $(\ldots 4)\ldots 8)$ and $(\ldots 6)$ is the corresponding decimal representation of 2^n, we have:

$$2^{n+4} = 2^n.16 = \begin{cases} (\ldots 2) \times 16 = (\ldots 2) \text{ because } 2 \times 6 = 12 \\ (\ldots 4) \times 16 = (\ldots 4) \text{ because } 4 \times 6 = 24 \\ (\ldots 8) \times 16 = (\ldots 8) \text{ because } 8 \times 6 = 48 \\ (\ldots 6) \times 16 = (\ldots 6) \text{ because } 6 \times 6 = 36 \end{cases}$$

That shows that the last digits appear periodically 2, 4, 8, 6 in any group of four consecutive powers of 2 from the beginning.

Now, let us have a look at the sequence **2** $\rightarrow$ **1, 5, 9, 13, 17, ...** (powers of 2^y ending on 2) and let us try to find a relation between **y** and the number **x** of the group of four corresponding powers of 2:

y=1 appears in the group **x=1**

y=5 appears in the group **x=2**

y=9 appears in the group **x=3**

y=13 appears in the group **x=4**

Thus we obtain that $\mathbf{y = 4x - 3 = 4(x - 1) + 1}$. Since this is true for $\mathbf{x = 1, 2, 3}$, let us suppose that for $\mathbf{y = 4x - 3}$, 2^y ends on **2** (i.e. **y** is in **(2)**). Then, $2^{4(x+1)-3} = 2^{4x+4-3} = 2^{4x-3} \times 16$ and since 2^{4x-3} ends on 2, $2^{4x-3} \times 16$ also ends on 2. Similarly we get that:

(2) $\rightarrow$ **1, 5, 9, 13, 17, ...** $\rightarrow$ $\mathbf{y = 4x - 3}$

(4) $\to$ ***2, 6, 10, 14, 18, ...*** $\to$ $\mathbf{y = 4x - 2}$
$\underline{\textbf{(8)} \to \textbf{3, 7, 11, 15, 19, ...}}$ $\to$ $\mathbf{y = 4x - 1}$
$\underline{\textbf{\textit{(6)}} \to \textbf{\textit{4, 8, 12, 16, 20, ...}}}$ $\to$ $\mathbf{y = 4x}$

(which does not come as a surprise if we have in mind modular arithmetic). Hence, the members of any of the four sequences can be written as $y_x = y_1 + 4(x-1)$ where y_1 belongs to **(2)**, ***(4)***, $\underline{\textbf{(8)}}$, or $\underline{(6)}$, correspondingly.

Now, let us try to find another interpretation of the interrelation between **y** and **x** – as coordinates of points in the xOy plane. We introduce the following notation:

$A_1^2(1,1)$, $A_2^2(2,5)$, $A_3^2(3,9)$, $A_4^2(4,13)$, ..., $A_n^2(n,y)$, ... where the superscript shows the last digit of 2^y, the subscript shows the group **x** to which 2^y belongs and (n,y) are the coordinates in the xOy plane. We can ask whether all these points belong to the same line, i.e. whether we have that $\mathbf{y = mx + b}$. We already know that in the group **(2),** $\mathbf{y = 4x - 3}$ and we indeed get that

$$m = \frac{4(x+1) - 3 - (4x-3)}{x+1-x} = 4$$

and accordingly we find $\mathbf{b = -3}$.

Similarly, we notice that the corresponding (x, y) in the other groups belong to the lines $\mathbf{y = 4x - 2}$, $\mathbf{y = 4x - 1}$, and $\mathbf{y = 4x}$, respectively, i.e. the points (x, y) belong to four parallel lines!

Having in mind the above observations, we can now give the solutions of the first two problems stated at the beginning:

Solution to Problem 1: 2011 must be the ordinate of a point on the line $\mathbf{y = 4x + b}$ (where **b** can be only 0, −1, −2 or −3). I.e. $\mathbf{2011 = 4x + b}$, so $x = \dfrac{2010+1}{4} - \dfrac{b}{4} = \dfrac{2014}{4} - \dfrac{3}{4} - \dfrac{b}{4} = 53 - \dfrac{b+3}{4}$, hence $\mathbf{b = -3}$, i.e. 2^{2011} ends on 2.

Solution to Problem 2: The power **n** in question should be in the sequence $\underline{\textbf{(8)} \to \textbf{3, 7, 11, 15, ...}}$ $\to$ $\mathbf{y = 4x - 1}$, and hence, we easily calculate that $\mathbf{y = 4.33 - 1 = 131}$. So, 2^{2011} is the 33rd power of 2 ending on 8.

We can conclude without any difficulty that what we have done for 2^n is true for a^n where **a** is any natural number having the last digit 2 in its decimal representation (i.e. 12^n, 22^n, ..., 532^n, ...), hence, we get easily:

Solution to Problem 3: For 372^{45}, $\mathbf{y = 45 = 4.12 - 3}$, hence **y** is in the sequence **(2)** $\to$ **1, 5, 9, 13, 17, ...** $\to$ $\mathbf{y = 4x - 3}$, so 372^{45} ends on 2.

We can generalize the same idea and find that the last digits in the decimal representation of 3^n periodically repeat in groups of four: 3, 9, 7,1 and that both 3^n and 3^{n+4} end on the same digit. Then, again we have that if 3^{4x-3} ends on 3, then $3^{4(x+1)-3}$ will end on 3 etc.

More generally, for a^n where **a** is a natural number ending on 3 in its decimal representation the same pattern as for 3^n hold.

Looking now on 4^y we have only groups of numbers ending on 4 and 6.

On the other hand obviously all natural powers of 1^y, 5^y, 6^y and 10^y end on 1, 5, 6 and 0, respectively.

For 7^y we have groups of 7, 9, 3, 1.

For 8^y we have groups of 8, 4, 2, 6.

For 9^y we have two groups of 9, 1.

In conclusion we can state the following:

Claim:

(1) If for $a \in N$, a^2 has the same last digit in its decimal representation as **a**, then for all natural **n**, a^n has the same last digit as **a**.

(2) If the last digits of **a** and a^2 are different and a^3 has the same last digit in its decimal representation as **a**, then for all natural **n**, the last digit of a^n repeats in a groups of two (namely the last digits of **a** anda^2).

(3) If the last digits of **a**, a^2 and $a^3$are different then the last digit of a^4 is different from all three last digits and the last digit of a^n repeats in a groups of four.

(4) For all $a \in N$, a^{n+4} has the same last digit as a^n.

The proof is inductive and easily done.

On the basis of the above observations many similar problems to those that we have considered can be formulated and solved by the students themselves.

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ДЕМОНСТРИРАНЕ КАК СЕ РАЖДА ЕДНА ИДЕЯ НА УЧЕНИЦИТЕ В СРЕДНОТО УЧИЛИЩЕ

Петра Генчова Стайнова

Предлага се един начин на запознаване на учениците в средното училище със задаването на общо математическо твърдение на основата на наблюдения и изводи на конкретен проблем от теория на числата.