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ABOUT HOMOGENEOUS SPACES AND CONDITIONS OF COMPLETENESS OF SPACES*

Alexander Arhangel'skii, Mitrofan Choban#, Ekaterina Mihaylova#

In this paper we introduce new notions of *o*-homogeneous space, *lo*-homogeneous space, *do*-homogeneous space and, *co*-homogeneous space. If a *lo*-homogeneous space X is first-countable at some point, then X is first-countable. If a *lo*-homogeneous space X contains a dense extremally disconnected subspace, then X is extremally disconnected.

1. Introduction. By a space we understand a Tychonoff topological space. We use the terminology from [7].

In this article we introduce new notions of homogenity of spaces: *o*-homogeneous space, *lo*-homogeneous space, *do*-homogeneous space and *co*-homogeneous space. Let $\mathcal{P}$ be one of the properties {*fan–complete, q–complete, sieve–complete*} and U be an open non-empty subset of X with the property $\mathcal{P}$. We prove that if X is a regular *lo*-homogeneous space and U is an open non-empty subset of X with the property $\mathcal{P}$, then X is a space with the property $\mathcal{P}$. Other properties of homogeneous spaces are studied as well.

2. Various types of completeness. In this section we list some several notions introduced in [2, 4].

For a sequence $\{H_n : n \in \omega\}$ of subsets of a space X, $Lim\{H_n : n \in \omega\}$ is the set of all accumulation points of $\{H_n : n \in \omega\}$. If $H_{n+1} \subseteq H_n$ for any $n \in \omega$, then $Lim\{H_n : n \in \omega\} = \cap\{cl_X H_n : n \in \omega\}$.

A sequence $\{U_n : n \in \omega\}$ of open subsets of a space X is called a *stable sequence* if it satisfies the following conditions:

(S1) $\emptyset \neq U_{n+1} \subseteq U_n$ for any $n \in \omega$;

(S2) Every sequence $\{V_n : n \in \omega\}$ of open non-empty sets in X such that $V_n \subseteq U_n$ for each $n \in \omega$, has an accumulation point in X, i.e. $Lim\{V_n : n \in \omega\} \neq \emptyset$.

A subset L of a space X is *bounded* if for every locally finite family γ of open subsets of X, the set $\{U \in \gamma : U \cap L \neq \emptyset\}$ is finite.

From conditions (S1) and (S2) it follows that $H = \cap\{cl_X U_n : n \in \omega\} = Lim\{U_n : n \in \omega\}$ is a bounded non-empty subset of X.

Let X be a space , $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ be a sequence of families of open subsets of X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of mappings.

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A sequence $\alpha = \{\alpha_n : n \in \omega\}$ is called a *c-sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every n. Let $H(\alpha) = \cap\{U_{\alpha_n}; n \in \omega\}$. Consider the following conditions:

(SC1) $\cup\{U_\beta : \beta \in A_n\}$ is a dense subset of X for each $n \in \omega$.

(SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ is a dense subset of the set U_α for all $\alpha \in A_n$ and $n \in \omega$.

(SC3) $X = \cup\{U_\mu : \mu \in A_1\}$ and $U_\alpha = \cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all $\alpha \in A_n$ and $n \in \omega$.

(SC4) $\cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$.

(C1) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is stable.

(C2) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, each sequence $\{y_n \in U_{\alpha_n}; n \in \omega\}$ has an accumulation point in X.

(C3) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is a base of open neighbourhoods of the set $H(\alpha)$ in X.

(C4) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the set $H(\alpha)$ is a non-empty compact subset of X.

(C5) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the set $H(\alpha)$ is a non-empty countably compact subset of X.

Sequences γ and π are called an A-sieve if they have the Properties (SC3), (SC4). They are called a dense A-sieve if they have the Properties (SC1), (SC2), (SC4).

A space X is called *densely sieve-complete* if there exists a dense A-sieve with the Properties (C2) and (C4). A space X is called *sieve-complete* if there exists an A-sieve with the Properties (C2) and (C4).

A space X is called *densely q-complete* if there exists a dense A-sieve with the Property (C2). A space X is called *q-complete* if there exists an A-sieve with the Properties (C2) and (C5).

A space X is called *densely fan-complete* if there exists a dense A-sieve on X with the Property (C1). A space X is called *fan-complete* if there exists an A-sieve on X with the Property (C1).

The sieve-complete and q-complete spaces were studied in [5, 6, 8, 10, 9]. The other classes of spaces were introduced in [2, 3, 4].

3. Some new generalization of homogenity. A space X is called:

– *o-homogeneous* if for any two points $a, b \in X$ there exists a continuous open mapping $h_{ab} : X \longrightarrow X$ such that $h_{ab}(a) = b$;

– *lo-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X and a continuous open mapping $h_{ab} : U \longrightarrow V$ such that $a \in U$, $b \in V$ and $h_{ab}(a) = b$;

– *co-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X and a continuous open mapping $h_{ab} : U \longrightarrow V$ such that $a \in U$, $b \in V$, $h(a) = b$ and the set $cl_X h_{ab}^{-1}(x)$ is countably compact for each $x \in V$;

– *do-homogeneous* if for any two points $a, b \in X$ there exist two open subsets U and V of X, two subsets A and B and a continuous open mapping $h_{ab} : A \longrightarrow B$ such that $a \in A \subseteq U \subseteq cl_X A$, $b \in B = h_{ab}(A) \subseteq V \subseteq cl_X B$ and $h_{ab}(a) = b$.

We mention that the space X is called *d-homogeneous* [1] if for any two points $a, b \in X$ there exist two dense subspaces A and B of X and a homeomorphism $h_{ab} : A \longrightarrow B$ such that $a \in A$, $b \in B$ and $h_{ab}(a) = b$.

Theorem 3.1 (see [1], Theorem 2.1)**.** *Let X be a regular locally separable first-countable space without isolated points. Then, X is d-homogeneous.*

Proof. Fix two points $a, b \in X$. There exists two open subsets U and V of X such that U and V are separable subspaces of the space X, $a \in U$ and $b \in V$. Fix a dense countable subset L of U and a dense countable subset M of V. We can assume that $a \in L$ and $b \in M$. The spaces L and M are homeomorphic to the space of rationals $\mathbb{Q}$. Hence, there exists a homeomorphism $g : L \longrightarrow M$ of L onto M such that $g(a) = b$.

We put $C = X \setminus cl_X(U \cup V)$, $A = B = L \cup M \cup C$, $g(x) = x$ for $x \in C$, $g(x) = h(x)$ for $x \in L$ and $g(x) = h^{-1}(x)$ for $x \in M$. The set $A = B$ is dense in X and $h : A \longrightarrow A$ is a homeomorphism. The proof is complete. □

Theorem 3.2 (see [1], Theorem 2.2)**.** *Let X be a regular do-homogeneous space. Then, $\chi(X) = \chi(x, X) = \chi(y, X)$ for any points $x, y \in X$.*

Proof. Fix two points $a, b \in X$. Then, there exist two open subsets U and V of X, two subsets A and B and a continuous open mapping $h : A \longrightarrow B$ such that $a \in A \subseteq U \subseteq cl_X A$, $b \in B = h(A) \subseteq V \subseteq cl_X B$ and $h(a) = b$. Then, $\chi(a, X) = \chi(a, A) = \chi(b, B) = \chi(b, X)$. The proof is complete. □

Theorem 3.3. *Let X be a regular lo-homogeneous space, $\mathcal{P}$ be one of the properties $\{fan-complete, q-complete, sieve-complete\}$ and U be an open non-empty subset of X with property $\mathcal{P}$. Then, X is a space with property $\mathcal{P}$.*

Proof. An open continuous image of a space with property $\mathcal{P}$ has property $\mathcal{P}$ (see [2, 3, 4, 5, 6]). Hence, X has locally property $\mathcal{P}$. Obviously, a space X has property $\mathcal{P}$ if and only if it has locally property $\mathcal{P}$. The proof is complete. □

Example 3.4. Let Y_0 be the space of irrationals, C be the unit circle, $Y_1 = C \times Y_0$ and X be the discrete sum of the spaces Y_0 and Y_1. The space X is not homogeneous, is *co*-homogeneous and for any two points $a, b \in X$ there exists an open continuous mapping h_{ab} with compact fibers of X onto X such that $h_{ab}(a) = b$.

Example 3.5. Let Y_0 be the space of irrationals, $\mathbb{R}$ be the space of reals, $Y_1 = \mathbb{R} \times Y_0$ and X be the discrete sum of the spaces Y_0 and Y_1. The space X is not *co*-homogeneous, it is *o*-homogeneous and for any two points $a, b \in X$ there exists an open continuous mapping h_{ab} of X onto X such that $h_{ab}(a) = b$.

Example 3.6. Let C be the unit circle, $\mathbb{R}$ be the space of reals and X be the discrete sum of the spaces C and $\mathbb{R}$. The space X is not *co*-homogeneous, not *o*-homogeneous but it is *lo*-homogeneous.

4. Extremal disconnectedness and *do*-homogenity of spaces. A space X is extremally desconnected if the closure of every open subset of X is open [7].

A point $x \in X$ is a point of extremal disconnectedness of the space X if $x \notin cl_X U \cap cl_X V$ for any two disjoint open subsets U and V of X. Obviously, the space X is extremally disconnected if and only if any point of X is a point of extremal disconnectedness of the space X. Moreover, if $x \in Y \subseteq X$ and Y is a dense subspace of the space X, then x is a point of extremal disconnectedness of the space X if and only if x is a point of extremal disconnectedness of the space Y.

Theorem 4.1. *Let X be a regular do-homogeneous space and $a \in X$ be a point of extremal disconnectedness of the space Y. Then, X is an extremally disconnected space.*

Proof. Fix a point $b \in X$. Then, there exist two open subsets U and V of X, two subsets A and B and a continuous open mapping $h : A \longrightarrow B$ such that $a \in A \subseteq U \subseteq cl_X A$, $b \in B = h(A) \subseteq V \subseteq cl_X B$ and $h(a) = b$. Then, a is a point of extremal disconnectedness of the space A and b is a point of extremal disconnectedness of the space B and of the space X. Hence, any point of X is a point of extremal disconnectedness. The proof is complete. □

Corollary 4.2 (see [1], Theorem 2.3)**.** *Let X be a regular do-homogeneous space and Y be a dense extremally disconnected subspace of the space X. Then, X is an extremally disconnected space.*

Corollary 4.3 (see [1], Theorem 3.4)**.** *Let X be a regular space, $X \times X$ be a do-homogeneous space and Y be a dense extremally disconnected subspace of the space $X \times X$. Then, X is a discrete space.*

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Alexander Arhangel'skii
33, Kutuzovskii prospekt
Moscow 121165, Russia
e-mail: arhangel.alex@gmail.com

Ekaterina Mihaylova
St. Kliment Ohridski University of Sofia
5, James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: katiamih@fmi.uni-sofia.bg

Mitrofan Choban
Department of Mathematics
Tiraspol State University
5, Iablochikin
MD 2069, Kishinev, Republic of Moldova
e-mail: mmchoban@gmail.com

ОТНОСНО ХОМОГЕННИ ПРОСТРАНСТВА И УСЛОВИЯ ЗА ПЪЛНОТА

Александър В. Архангелски, Митрофан М. Чобан, Екатерина П. Михайлова

Въведени са понятията o-хомогенно пространство, lo-хомогенно пространство, do-хомогенно пространство и co-хомогенно пространство. Показано е, че ако lo-хомогенно пространство X има отворено подпространство, което е q-пълно, то и самото X е q-пълно. Показано е, че ако lo-хомогенно пространство X съдържа навсякъде гъсто екстремално несвързано подпространство, тогава X е екстремално несвързано.