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ABOUT HOMOGENEOUS SPACES AND THE BAIRE PROPERTY IN REMAINDERS*

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In this paper we continue the study of the notions of *o*-homogeneous space, *lo*-homogeneous space, *do*-homogeneous space and *co*-homogeneous space. Theorem 5.1 affirms that a *co*-homogeneous space X is a Moscow space provided it contains a G_δ-dense Moscow subspace Y.

1. Introduction. By a space we understand a Tychonoff topological space. We follow the terminology given in [15]. The present paper is a continuation of the article [1], which contains the definitions of *o*-homogeneous space, fan-complete space, *q*-complete space, sieve-complete space, *lo*-homogeneous space, *do*-homogeneous space and *co*-homogeneous space. A *remainder* of a space X is the subspace $Y \setminus X$ of a Tychonoff extension Y of X. The space Y is an extension of X if X is a dense subspace of Y. In this article we investigate what kind of remainders a space can have.

Problem A. *Let $\mathcal{P}$ be a property and Y be an extension of a space X. Under which conditions the remainder $Y \setminus X$ has the property $\mathcal{P}$?*

In [2, 3, 4, 5, 6, 7] Problem A was studied for topological groups. Some results for rectifiable spaces were obtained in [11].

2. On *o*-homogeneous spaces and Moscow spaces. A space X is called *a Moscow space* [8] if for each open subset U of X the closure $cl_X U$ of U in X is the union of a family of G_δ-subsets of X.

The following fact for *d*-homogeneous spaces was proved in [9].

Theorem 2.1. *Let Y be a G_δ-dense subspace of a co-homogeneous space X. If Y is a Moscow space, then X is a Moscow space and Y is C-embedded in X.*

Proof. First we prove the next statement.

Claim 1. If U is an open subset of X and $y \in Y \cap cl_X U$, then there exists a G_δ-subset H of X such that $y \in H \subseteq cl_X U$.

Indeed, since Y is a Moscow space, there exists a G_δ-subset H of X such that $y \in H \cap Y = cl_Y(U \cap Y) \subseteq cl_X U$. We affirm that $H \subseteq cl_X U$. Indeed, assume the contrary. Then, $P = H \setminus cl_X U$ is a non-empty G_δ-subset of the space X and $P \cap Y = \emptyset$. Since Y is G_δ-dense in X, we have $P \cap Y \neq \emptyset$, a contradiction. The Claim 1 is proved.

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Now we fix an open subset V of the space X. Fix a point $b \in Y$. Let $a \in cl_X V$. Fix two open subsets U_1, U_2 of X and an open continuous mapping $h : X \longrightarrow X$ such that $b \in U_1$, $a \in U_2$, $h(b) = a$ and the set $cl_X h^{-1}(x)$ is countably compact for each $x \in U_2$. We put $W = V \cap U_2$ and $U = h^{-1}(W)$. By construction, $a \in cl_X W$ and $b \in cl_X U$. By virtue of Claim 1, there exists a G_δ-subset H of X such that $b \in H \subseteq cl_X U$. Fix a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of X such that $cl_X V_{n+1} \subseteq V_n \cap U_1$ for each $n \in \mathbb{N}$ and $b \in \cap\{V_n : n \in \mathbb{N}\} \subseteq H$. Then, $P = \cap\{h(V_n) : n \in \mathbb{N}\} = h(\cap\{V_n : n \in \mathbb{N}\})$ and $a \in P \subseteq cl_X W$. Hence, $cl_X U$ is the union of a family of G_δ-subsets of X.

In addition, every G_δ-dense subspace Y of a Moscow space X is C-embedded in X [16]. The proof is complete. $\square$

3. The Baire property in remainders of spaces. Let X be a space. We put $lc(X) = \cup\{U : cl_X U$ *is a compact subspace*$\}$ and $b(X) = \cup\{U : U$ *is an open subspace with the Baire property*$\}$. Obviously, $lc(X) \subseteq b(X)$. If $rb(X) = X \setminus b(X)$, then $rb(X)$ is a subset of the first category, $b(X)$ is a subspace with the Bare property and $X = b(X) \cup rb(X)$. A space X is without the Baire property if and only if $rb(X) \neq \emptyset$.

Proposition 3.1. *Let Y be an extension of a space X and the space $Z = Y \setminus X$ be without Baire property. Then, there exist an open non-empty subset U of X and a G_δ-subset S of Y such that $S \subseteq U \subseteq cl_X S$.*

Proof. If $lc(X) \neq \emptyset$, then $U = S = lc(X)$ is an open non-empty subset of the spaces X and Y. In this case the assertions of Proposition 3.1 are obvious.

Suppose that $lc(X) = \emptyset$, i.e. the space X is nowhere locally compact.

Let $V = X \setminus cl_Y(Y \setminus X)$. If $V \neq \emptyset$, then we can assume that $U = S = V$. In this case the assertions of Proposition 3.1 are obvious too.

Suppose now that $V = \emptyset$ and $rb(Z) \neq \emptyset$. In this case the space Z is dense in Y. There exists a sequence $\{Z_n : n \in \omega = \{1, 2, \dots\}\}$ of closed nowhere dense subsets of the space Z such that $rb(Z) = \cup\{Z_n : n \in \omega\}$. Fix an open non-empty subset W of Y such that $W \cap Z = Z \setminus cl_Y b(Z)$. Any set $F_n = cl_Y Z_n$ is nowhere dense in Y. We put $U = W \cap X$ and $S = W \setminus \cup\{F_n : n \in \omega\}$. The proof is complete. $\square$

It is well known that a dense G_δ-subspace of a densely fan-complete space is a densely fan-complete space, a G_δ-subspace of a densely q-complete space is a densely q-complete space, a space is densely fan-complete if and only if it contains a dense fan-complete subspace and a space is densely sieve-complete if and only if it contains a dense paracompact Čech-complete subspace (see [10, 11, 12]).

Thus from Proposition 3.1 it follows.

Corollary 3.2. *Let Y be a densely fan-complete extension of a space X. Then, either the remainder $Z = Y \setminus X$ has the Baire property, or there exist an open non-empty subset U of X and a G_δ-subset S of Y such that $S \subseteq U \subseteq cl_X S$ and S is a fan-complete subspace.*

Corollary 3.3. *Let Y be a densely q-complete extension of a space X. Then, either the remainder $Z = Y \setminus X$ has the Baire property, or there exists an open non-empty subset U of X such that U is a densely q-complete subspace.*

Corollary 3.4. *Let Y be a densely sieve-complete extension of a space X. Then, either the remainder $Z = Y \setminus X$ has the Baire property, or there exist an open non-empty* *subset U of X and a G_δ-subset S of Y such that $S \subseteq U \subseteq cl_X S$ and S is a paracompact Čech-complete subspace.*

4. On dissentive spaces. A *dissentive* operation (a *dissentor*) on a space X (see [11]) is a continuous mapping $\mu : X^3 \to X$ satisfying the following conditions:

– $\mu(x, x, y) = y$ for all $x, y \in X$;

– for every open set U of X and all $b, c \in X$, the set $\mu(U, b, c) = \{\mu(x, b, c) : x \in U\}$ is open in X.

A space is *dissentive* if it admits a dissentive operation. A dissentive operation μ is a Mal'cev dissentive operation if $\mu(x, y, y) = x$ for all $x, y \in X$.

A *rectification* on a space X is a homeomorphism $\varphi : X \times X \to X \times X$ with the following two properties:

– $\varphi(\{x\} \times X) = (\{x\} \times X)$, for every $x \in X$;

– there exists $e \in X$ such that $\varphi(x, x) = (x, e)$ for every point $x \in X$.

The point $e \in X$ is called *neutral element* of the space X. A space with a rectification is called *a rectifiable* space.

A homogeneous algebra on a space G is a pair of binary continuous operations $p, q : G \times G \to G$ such that $p(x, x) = p(y, y)$, and $p(x, q(x, y)) = y$, $q(x, p(x, y)) = y$ for all $x, y \in G$ (see [14], [13]). A space is rectifiable if and only if it admits a structure of the homogeneous algebra (see [14], [13]).

Every rectifiable space is dissentive. If $p, q : G \times G \to G$ is a structure of homogeneous algebra on a space G, then $\mu(x, y, z) = p(x, q(y, z))$ is a Mal'cev dissentive operation.

A space X is called *weight homogeneous* if there exists a cardinal number τ such that the family $\{U : w(U) = \tau, U \textit{ is open in } X\}$ is a base of X.

Every dissentive space is o-homogeneous and any lo-homogeneous space is weight homogeneous.

Lemma 4.1. *Let U be an open non-empty densely fan-complete subspace of an lo-homogeneous space X. Then, X is densely fan-complete.*

Proof. Fix a point $a \in U$. For any point $b \in X$ we fix two open subsets V and W and an open continuous mapping $h_{ab} : V \to W$ such that $a \in V \subseteq U$, $b \in W$ and $h_{ab}(a) = b$. We put $V_b = h_{ab}(V)$. Since V_b is an open continuous image of the densely fan-complete space V, V_b is a densely fan-complete space (see [12]). Fix on X a well-ordering. We put $W_b = V_b \setminus cl_X(\cup\{V_c : c < b\})$. Any subspace W_b is densely fan-complete. Hence, $H = \cup\{V_c : c < b\}$ is an open densely fan-complete subspace. Therefore, X is a densely fan-complete space. □

The proofs of the next two lemmas are similar.

Lemma 4.2. *Let U be an open non-empty densely q-complete subspace of an lo-homogeneous space X. Then, X is densely q-complete.*

Lemma 4.3. *Let U be an open non-empty densely sieve-complete subspace of an lo-homogeneous space X. Then, X contains some dense paracompact Čech-complete subspace.*

Theorem 4.4. *Let Y be a densely fan-complete extension of an lo-homogeneous space X. Then, if $Z = Y \setminus X$ is a space without the Baire property, then X is a densely fan-complete space.*

Proof. Suppose that $Z = Y \setminus X$ is a space without Baire property. By virtue of Proposition 3.1, there exist an open non-empty subset U of X and a G_δ-subset S of Y such that $S \subseteq U \subseteq cl_X S$. By construction, S and U are densely fan-complete subspaces of the space X. Lemma 4.1 completes the proof. □

The proof of the following theorem is similar, by using Lemma 4.2.

Theorem 4.5. *Let Y be a densely q-complete extension of an lo-homogeneous space X. Then, if $Z = Y \setminus X$ is a space without the Baire property, then X is a densely q-complete space.*

From Proposition 4.1 it follows

Corollary 4.6. *Let Y be an extension of an lo-homogeneous space X and Y has the Baire property. Then, either the remainder $Z = Y \setminus X$ has the Baire property, or X is a space with the Baire property.*

Theorem 4.7. *Let Y be a densely sieve-complete extension of an lo-homogeneous space X and X is nowhere locally compact. Then, the following assertions are equivalent:*
1. *X is a space of the first category.*
2. *$Z = Y \setminus X$ is a densely sieve-complete space.*

Proof. If X is a space of the first category, then there exists a sequence of open dense in Y sets $\{U_n : n \in \omega\}$ such that $L = \cap\{U_n : n \in \omega\} \subseteq Z$. Thus L and Z are densely sieve-complete spaces. The implication $1 \to 2$ is proved.

Let Z be a densely sieve-complete space. Then, Z contains some dense Čech-complete subspace L. In this case $X \supseteq Y \setminus L$ and $Y \setminus L$ is a subset of the first category in Y. The proof is complete. □

Corollary 4.8. *Let Y be a densely sieve-complete extension of an lo-homogeneous space X and X is nowhere locally compact. Then, the following assertions are equivalent:*
1. *$Z = Y \setminus X$ is a space of the first category.*
2. *X is a densely sieve-complete space.*

Corollary 4.9. *Let Y be a densely sieve-complete extension of a lo-homogeneous space X. Then either the remainder $Z = Y \setminus X$ has the Baire property, or X is a densely sieve-complete space.*

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ОТНОСНО ХОМОГЕННИ ПРОСТРАНСТВА И СВОЙСТВОТО НА БЕР В ПРИРАСТА

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В съобщението е продължено изследването на понятията o-хомогенно пространство, lo-хомогенно пространство, do-хомогенно пространство и co-хомогенно пространство. Показано е, че ако co-хомогенното пространство X съдържа G_δ-гъсто Московско подпространство, тогава X е Московско пространство.