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ABOUT TOPOLOGICAL GROUPS AND THE BAIRE PROPERTY IN REMAINDERS*

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In this paper we study the remainders with Baire property of topological groups.

1. Introduction. By a space we understand a Tychonoff topological space. We use the terminology from [11]. The present paper is a continuation of the articles [1, 2], which contain the definitions of an *o*-homogeneous space, a fan-complete space, a *q*-complete space, a sieve-complete space, an *lo*-homogeneous space, a *do*-homogeneous space, a *co*-homogeneous space.

A *remainder* of a space X is the subspace $Y \setminus X$ of a Tychonoff extension Y of X. The space Y is an extension of X if X is a dense subspace of Y.

In this article we consider what kind of remainders can have a space.

Problem A. *Let $\mathcal{P}$ be a property and Y be an extension of a space X. Under which conditions the remainder $Y \setminus X$ has the property $\mathcal{P}$?*

In [3, 4, 5, 6, 7, 8] the Problem A was examined for topological groups. Some results for rectifiable spaces were obtained in [9].

A particular case of the Problem A is the next question

Question A. *Under which conditions the Stone-Čech compactification βX of a space X is the Stone-Čech compactification of the remainder $\beta X \setminus X$?*

In [4], Theorem 1.1, A. V. Arhangel'skii has proved: If a topological group G is a dense subspace of the Čech-complete space X and G is not Čech-complete, then the subspace $Y = X \setminus G$ is dense in X and has the Baire property. We establish that analogous result is true for a more large classes of spaces.

One of the first remarcable results concerning Problem A was obtained by E. Čech, M. Henriksen and J. R. Isbel. Theorem of E. Čech affirms that for any space X the character $\chi(x, \beta X)$ of any point $x \in \beta X \setminus X$ is uncountable. Theorem of M. Henriksen and J. R. Isbel affirms that the remainder $\beta X \setminus X$ is a Lindelöf space if and only if the space X is of countable type (see [11, 12]).

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2. On remainders of topological groups. Let G be a topological group. Let ρG be the Raikov completion of a topological group G. If the topological group G is densely fan-complete, then the Raikov completion ρG is a paracompact Čech-complete space.

A space X is called a *paracompact p-space* if it admits a perfect mapping onto a metrizable space. A feathered group is a topological group whose underlying space is a paracompact p-space. A topological group is a feathered group if and only if it is a space of pointwise countable type (see [10]).

In [9] it was proved that a topological group G is densely fan-complete if and only if it is fan-complete and, consequently, G is a dense G_δ-subspace of some pseudocompact space. Any topological group is an o-homogeneous space. Thus from Theorem 4.4 in [2] it follows:

Corollary 2.1. *Let Y be a densely fan-complete extension of a topological group G. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or G is a fan-complete space.*

In [9] it was proved that a topological group G is densely q-complete if and only if it is q-complete and there exits a countably compact subgroup H such that the quotient space G/H is metrizable and the projection $\pi : G \to G/H$ is open and closed. Moreover, a q-complete topological group is a G_δ-subset in some countably compact extension. Thus from Theorem 4.5 in [2] it follows:

Corollary 2.2. *Let Y be a densely q-complete extension of a topological group G. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or G is a q-complete space.*

From Theorem 4.5 in [2] it follows

Corollary 2.3. *Let Y be a densely sieve-complete extension of a topological group G. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or G is a paracompact Čech-complete space.*

Corollary 2.4 (A. V. Arhangel'skii [4])**.** *Let Y be a Čech-complete extension of a topological group G. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or G is a paracompact Čech-complete space.*

Example 2.5. Let $\xi \in \beta\omega \setminus \omega$ and $X = \{\xi\} \cup \omega$. We put $L = \mathbb{R}^X$ and $B = C_p(X) \subseteq L$. D. J. Lutzer and R. A. McCoy [13] proved that the space B is not complete metrizable and has the Baire property. Since B is not Čech-complete, from Corollary 2.3 (or from Theorem 1.1 of [4]) it follows that $Y = L \setminus B$ is a dense subspace of L and has the Baire property. Thus the linear space L is a complete metrizable extension of the spaces B, Y with the Baire property, B is a linear subspace and it is not complete metrizable.

3. Embedding into remainders of topological groups. The following fact is a generalization of Theorem 2.18 from [8].

Theorem 3.1. *Let Y be a space. Then there exist a compact abelian group A and a dense subgroup B of A such that:*

1. *$X = A \setminus B$ is a pseudocompact space.*
2. *Y is a closed subspace of the space X.*
3. *A is a compactification of the space X.*

Proof. There exists a compact space K such that Y is a nowhere dense subspace of the space K and the subspace $\Phi = K \setminus cl_K Y$ is not paracompact and has not G_δ-points.

Fix a point $e \in \Phi \subseteq K$. Then, there exists a compact abelian group A with the properties:

1. K is a subspace of the space A and e is the unity of the group A.

2. For any continuous mapping $f : K \longrightarrow H$ into a compact abelian group H for which $f(e)$ is the unity of H there exists a continuous homomorphism $\bar{f} : A \longrightarrow H$ such that $f = \bar{f}|B$.

3. The group G algebraically generated by the set K in A is dense in A.

Put $Z = K \setminus Y$ and denote by B the subgroup of A algebraically generated by the set Z. Now, put $X = A \setminus B$. Let $F_1 = K \cup \{x^{-1} : x \in K\}$ and $F_n = F_1^n$ for each $n \in \mathbb{N}$. Then, $G = \cup\{F_n : n \in \mathbb{N}\}$ and each set F_n is nowhere dense in A.

By construction, the set B is dense in A. Thus each set $B_n = F_n \cap B$ is nowhere dense in B.

Claim 1. The subspace Y is closed in X.

The set K is compact and $Y = K \cap X$.

Claim 2. The group B is not locally compact.

If B is locally compact, then B is open-and-closed in A, that is a contradiction.

Claim 3. The space A is a compactification of the space X.

This assertion follows from Claim 2.

Claim 4. The space X is not Lindelöf.

Assume that the space X is Lindelöf. Since A is a compactification of the space B and $X = A \setminus B$, by virtue of Theorem of M. Henriksen and J. R. Isbel the space B is of countable type (see [11, 12]). Then, in B there exists a compact subgroup C of countable character such that $C \cap K \subseteq \Phi$. The natural projection $g : A \longrightarrow A/C$ is an open-and-closed continuous homomorphism and $g^{-1}(g(B)) = B$. Thus A, X and Y are Lindelöf p-spaces. Since C is compact, there exist $n \in \mathbb{N}$ and an open non-empty subset V of C such that $V \subseteq C \cap F_n$. Then we can assume that $C = V \subseteq F_n$. In this case $C \cap K = \{e\}$ and the space K has a countable base at the point e, a contradiction. Claim is proved.

Claim 5. The space X is pseudocompact.

Since A is not a paracompact p-space, the set X is G_δ-dense in A and X is pseudocompact [8, 9]. The proof is complete.

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ОТНОСНО ТОПОЛОГИЧНИ ГРУПИ И СВОЙСТВОТО НА БЕР В ПРИРАСТА

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