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SELF-AVOIDING WALKS IN THE PLANE*

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We examine the number of self-avoiding walks with a fixed length on the square grid graph and more specifically we complete the analysis of the lattice strip of height one. By combinatorial arguments we get an exact formula for the number of self-avoiding walks on a restricted to the left and to the right lattice strip. We investigate the formula asymptotically as well.

1. Introduction. A self-avoiding walk (*SAW* for convenience) is a path on a lattice, which does not intersect itself. In other words, if we consider it as a sequence of points $(a_1, a_2 \dots a_n)$, the following condition $a_i \neq a_j \ \forall i, j | i \neq j$ is satisfied. Finding the number of SAWs with a fixed length on the lattice $\mathbb{Z} \times \mathbb{Z}$ remains an open problem in combinatorics. However, the case $\mathbb{Z} \times \{0, 1\}$ is of interest, because it may lay the grounds for solving the general case.

Definition 1. *Let c_n denote the number of sequences $C = (c_0, c_1 \dots c_n) | c_i = (x_i, y_i)$ of pairwise different points in the plane such that $x_i \in \mathbb{Z}$ and $y_i \in \{0, 1\}$ for $0 \le i \le n$, $c_0 = (0, 0)$ and $|x_i - x_{i-1}| + |y_i - y_{i-1}| = 1$ for $1 \le i \le n$.*

Doron Zeilberger was the first who find a formula for c_n by using generating functions [1].

Theorem 1. *The following relation holds:*

$$c_n = 8 f_n - \sigma_n,$$

where $\sigma_n = \begin{cases} n & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd} \end{cases}$ *and f_n is the n^{th} Fibonacci number.*

Later Arthur Benjamin presented the first combinatorial proof of this result in [2] by generating SAWs from sequences, related to Fibonacci numbers. Nikolai Nikolov proved the formula for c_n by analyzing the construction of a SAW and counting the number of sequences in the different subsets [3].

A more complicated problem is considering the same grid $\mathbb{Z} \times \{0, 1\}$ with restrictions to the left and to the right.

Definition 2. *Let w_{abn} denote the number of sequences $C = (c_0, c_1 \dots c_n) | c_i = (x_i, y_i)$ of pairwise different points, such that $x_i \in \mathbb{Z} | -a \le x_i \le b$, $y_i \in \{0, 1\}$, $0 \le i \le n$.*

We study this number in order to complete the basic analysis of this lattice. In the proof we use the notions as follows: *u-move* for an upward move, *d-move* for a downward move, *l-move* for a move to the left and *r-move* for a move to the right. A sequence of the letters u, d, l and r is a SAW, which starts from $(0, 0)$ and follows the directions in the string. If a direction $x \in \{u, d, l, r\}$ is repeated k times we write x^k.

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2. Main result. ***2.1. Formula for restricted walks.*** For convenience we widen the definition of a binomial coefficient.

Definition 3. *We use the convention*

$$\binom{n}{k} = \begin{cases} 0 & \text{if } n < k \text{ or } n < 0 \text{ or } k < 0, \\ \dfrac{n!}{k!(n-k)!} & \text{if } n \geq k \geq 0. \end{cases}$$

We analyze the construction of the SAWs. In other words, in order to find w_{abn} we need to consider the different possible subsets of each sequence.

Definition 4. *Let l_{abn} and r_{abn} denote the number of SAWs C, such that the last move (c_{n-1}, c_n) is entirely on the left and entirely on the right of the segment $((0,0),(0,1))$, respectively. These sequences we call* left *and* right *SAWs.*

The relation $w_{abn} = l_{abn} + r_{abn}$ holds. Since the two cases are analogical, we proceed only with the case of the left sequences.

Definition 5. *Let fl_{an} denote the number of SAWs C, such that $x_i \leq 0$, $0 \leq i \leq n$ and pl_{abn} denote $l_{abn} - fl_{an}$, the number of the rest left sequences. These SAWs we call* fully left *and* partially left*, respectively.*

Definition 6. *Let ul_{an} denote the number of the fully left sequences $C | x_i - x_{i-1} + |y_i - y_{i-1}| = 1$, $1 \leq i \leq n$ and $ml_{an} = fl_{an} - ul_{an}$ is the number of the rest fully left SAWs. We call these sequences* ultra-left *and* middle left *SAWs, respectively.*

Definition 7. *Let ul_{ian} denote the number of the ultra-left sequences $C | x_n = -i$. We call these sequences* ultra-left *of type i.*

Proposition 1. *The following identity holds:*

$$\begin{aligned} w_{abn} &= \sum_{i=0}^{a} \binom{i+1}{n-i} + \sum_{i=0}^{a-2}\sum_{j=2}^{a-i} \binom{i+1}{n-2j-i} + \sum_{k=2}^{b+1}\sum_{i=0}^{a-1} \binom{i+1}{n-2k-i} \\ &+ \sum_{k=2}^{b+1}\sum_{i=0}^{a-3}\sum_{j=2}^{a-i-1} \binom{i+1}{n-2k-2j-i} + \sum_{i=0}^{b} \binom{i+1}{n-i} + \sum_{i=0}^{b-2}\sum_{j=2}^{b-i} \binom{i+1}{n-2j-i} \\ &+ \sum_{k=2}^{a+1}\sum_{i=0}^{b-1} \binom{i+1}{n-2k-i} + \sum_{k=2}^{a+1}\sum_{i=0}^{b-3}\sum_{j=2}^{b-i-1} \binom{i+1}{n-2k-2j-i} + \sigma_{abn}, \end{aligned}$$

$$\text{where } \sigma_{abn} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \max\left(0, \min\left(b, \dfrac{n-5}{2}\right)\right) - \max\left(0, \dfrac{n-2a-5}{2}\right) + \lambda_{an} & \\ + \max\left(0, \min\left(a, \dfrac{n-5}{2}\right)\right) - \max\left(0, \dfrac{n-2b-5}{2}\right) + \lambda_{bn} & \text{if } n \text{ is odd}, \end{cases}$$

$$\text{where } \lambda_{an} = \begin{cases} 0 & \text{if } n > 2a+1, \\ 1 & \text{if } n \text{ is odd and } n \leq 2a+1. \end{cases}$$

Proof. We consider an ultra-left sequence of type i and see that there are exactly i l-moves. The rest $n - i$ moves are either u-moves or d-moves. Each one of them is made after an l-move or in the beginning. Hence, a SAW is determined by the choice of the positions of the u-moves and d-moves. Thus

$$ul_{ian} = \binom{i+1}{n-i}.$$

We note the following relation:

$$ul_{an} = \sum_{i=0}^{a} ul_{ian} = \sum_{i=0}^{a} \binom{i+1}{n-i}. \tag{2.1}$$

Let us consider a middle left SAW $C = (c_0, c_1 \dots c_n)$. If n is odd and $n \geq 2a+1$, then the sequence $l^{\frac{n-1}{2}} ur^{\frac{n-1}{2}}$ is a SAW. The construction of the rest middle left walks involves an ultra-left sequence, followed by an u-turn of type $l^x ur^{x-1}$ or $l^x dr^{x-1}$. We associate C with an ultra-left sequence $C' = (c_0, c_1 \dots c_k) | x_k = x_n$ of type i, $k \leq n-3$. Considering the behavior of the u-turn $(c_{k-1}, c_k) \cup (C \backslash C')$ with the restriction coming from a, we obtain

$$ml_{an} = \sum_{i=0}^{a-2} \sum_{j=2}^{a-i} ul_{ia,(n-2j)} + \delta_{an} = \sum_{i=0}^{a-2} \sum_{j=2}^{a-i} \binom{i+1}{n-2j-i} + \delta_{an}, \tag{2.2}$$

where $\delta_{an} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd and } n > 2a+1, \\ 1 & \text{if } n \text{ is odd and } n \leq 2a+1. \end{cases}$

From the relation $fl_{an} = ul_{an} + ml_{an}$, (2.1) and (2.2) we get

$$fl_{an} = \sum_{i=0}^{a} \binom{i+1}{n-i} + \sum_{i=0}^{a-2} \sum_{j=2}^{a-i} \binom{i+1}{n-2j-i} + \delta_{an}. \tag{2.3}$$

The arbitrarily chosen partially left SAW $C = (c_0, c_1 \dots c_n)$ consists of a u-turn of type $r^x ul^{x+1}$ or $r^x dl^{x+1}$ followed by a fully left sequence. Since $x \geq 1$, having in mind the restriction deriving from b, we reach to the conclusion that

$$pl_{abn} = \sum_{k=2}^{b+1} fl_{(a-1),(n-2k)} = \sum_{k=2}^{b+1} \sum_{i=0}^{a-1} \binom{i+1}{n-2k-i} \tag{2.4}$$

$$+ \sum_{k=2}^{b+1} \sum_{i=0}^{a-3} \sum_{j=2}^{a-i-1} \binom{i+1}{n-2k-2j-i} + \sum_{k=2}^{b+1} \delta_{(a-1),(n-2k)}. \tag{2.5}$$

From the relation $l_{abn} = fl_{an} + pl_{abn}$, (2.3) and (2.4) follows that

$$l_{abn} = \sum_{i=0}^{a} \binom{i+1}{n-i} + \sum_{i=0}^{a-2} \sum_{j=2}^{a-i} \binom{i+1}{n-2j-i} + \delta_{an} + \sum_{k=2}^{b+1} \sum_{i=0}^{a-1} \binom{i+1}{n-2k-i} \tag{2.6}$$

$$+ \sum_{k=2}^{b+1} \sum_{i=0}^{a-3} \sum_{j=2}^{a-i-1} \binom{i+1}{n-2k-2j-i} + \sum_{k=2}^{b+1} \delta_{(a-1),(n-2k)}. \tag{2.7}$$

Now we consider the case of r_{abn} similarly. We swap the places of a and b in (2.5) to obtain the result. □

During the process of finding the formula, we used a computer program to verify the result.

2.2. Asymptotic estimations. Some special cases are of interest for our research, namely when one of the restrictions is removed. If we set $a = \infty$ and $b = \infty$, then the case coincides with c_n. So,

$$w_{\infty\infty n} = c_n.$$

If we set $a =$ const and $b = \infty$, then we obtain the formula

$$\begin{aligned} w_{a\infty n} &= \frac{1}{2}(a+1+(a+1)(-1)^{n+1}) + 4f_n - 2f_{n-2a-2} \\ &+ \sum_{i=0}^{a}\binom{i+1}{n-i} + \sum_{i=0}^{a-2}\sum_{j=2}^{a-i}\binom{i+1}{n-2j-i} + \delta_{an} + \sum_{k=2}^{b+1}\sum_{i=0}^{a-1}\binom{i+1}{n-2k-i} \\ &+ \sum_{k=2}^{b+1}\sum_{i=0}^{a-3}\sum_{j=2}^{a-i-1}\binom{i+1}{n-2k-2j-i} + \sum_{k=2}^{b+1}\delta_{(a-1),(n-2k)}. \end{aligned}$$

The form of the relations makes it convenient to estimate them asymptotically.

Proposition 2. *The following relations hold:*

$$\begin{aligned} w_{abn} &\in O(1) \\ w_{a\infty n} &\in O(q^n) \\ w_{\infty\infty n} &\in O(q^n), \end{aligned}$$

where $a =$ const, $b =$ const *and* $q = \dfrac{1+\sqrt{5}}{2}$.

Proof. We can choose k large enough, so that $w_{abn} = 0\ \forall n \geq k$. Hence, $w_{abn} \in O(1)$.

Let

$$\begin{aligned} v_{an} &= \frac{1}{2}(a+1+(a+1)(-1)^{n+1}) + 4f_n - 2f_{n-2a-2}, \\ o_{an} &= \sum_{i=0}^{a}\binom{i+1}{n-i} + \sum_{i=0}^{a-2}\sum_{j=2}^{a-i}\binom{i+1}{n-2j-i} + \delta_{an} + \sum_{k=2}^{b+1}\sum_{i=0}^{a-1}\binom{i+1}{n-2k-i} \\ &+ \sum_{k=2}^{b+1}\sum_{i=0}^{a-3}\sum_{j=2}^{a-i-1}\binom{i+1}{n-2k-2j-i} + \sum_{k=2}^{b+1}\delta_{(a-1),(n-2k)}. \end{aligned}$$

Then,

$$\begin{aligned} v_{an} &\in O\left(\frac{1}{2}\left(a+1+(a+1)(-1)^{n+1}\right) + 4f_n - 2f_{n-2a-2}\right) \\ &= O(4f_n - 2f_{n-2a-2}) \\ &= O(4f_n) \\ &= O(f_n) \\ &= O(q^n) \\ &\text{and} \\ o_{an} &\in O(1). \end{aligned}$$

Hence, $w_{a\infty n} \in O(q^n)$.

Similarly,

$$\begin{aligned} w_{\infty\infty n} &\in O(8f_n - \sigma_n) \\ &= O(8f_n) \\ &= O(f_n) \\ &= O(q^n). \end{aligned}$$

Therefore, $w_{\infty\infty n} \in O(q^n)$ to complete the proof. □

The limits which we derive in the three cases are as follows:

Proposition 3.

$$\begin{aligned} \lim_{n\to\infty} \frac{w_{abn}}{q^n} &= 0, \\ \lim_{n\to\infty} \frac{w_{a\infty n}}{q^n} &= \frac{1}{\sqrt{5}}\left(4 - \frac{2}{q^{2a+2}}\right), \\ \lim_{n\to\infty} \frac{w_{\infty\infty n}}{q^n} &= \frac{8}{\sqrt{5}}, \end{aligned}$$

where $a = \text{const}$, $b = \text{const}$ *and* $q = \dfrac{1+\sqrt{5}}{2}$.

Proof. We can choose k large enough, so that $w_{abn} = 0$ $\forall n \geq k$. Therefore, $\lim_{n\to\infty} \dfrac{w_{abn}}{q^n} = 0$.

Now

$$\begin{aligned} \lim_{n\to\infty} \frac{w_{a\infty n}}{q^n} &= \lim_{n\to\infty} \frac{v_{an}}{q^n} + \lim_{n\to\infty} \frac{o_{an}}{q^n} \\ &= \lim_{n\to\infty} \frac{4f_n - 2f_{n-2a-2}}{q^n} + 0 \\ &= \frac{1}{\sqrt{5}} \frac{4q^n - 2q^{n-2a-2}}{q^n}. \end{aligned}$$

Hence, $\lim_{n\to\infty} \dfrac{w_{a\infty n}}{q^n} = \dfrac{1}{\sqrt{5}}\left(4 - \dfrac{2}{q^{2a+2}}\right)$.

When $a = \infty$ and $b = \infty$

$$\begin{aligned} \lim_{n\to\infty} \frac{w_{\infty\infty n}}{q^n} &= \\ \lim_{n\to\infty} \frac{8f_n - \sigma_n}{q^n} &= \\ \frac{1}{\sqrt{5}} \frac{8q^n}{q^n}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \dfrac{w_{\infty\infty n}}{q^n} = \dfrac{8}{\sqrt{5}}$, which completes the proof. □

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НЕСАМОПРЕСИЧАЩИ СЕ РАЗХОДКИ В РАВНИНАТА

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Разглеждаме броя на несамопресичащите се разходки с фиксирана дължина върху целочислената решетка. Завършваме анализа върху случая за лента, с дължина едно. Чрез комбинаторни аргументи получаваме точна формула за броя на разходките върху лента, ограничена отляво и отдясно. Формулата я изследваме и асимптотично.