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CLASSIFICATION OF THE MAXIMAL SUBSEMIGROUPS OF THE SEMIGROUP OF ALL PARTIAL ORDER-PRESERVING TRANSFORMATIONS*

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Let PT_n be the semigroup of all partial transformations on an n - element set. A transformation $\alpha \in PT_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all x, y from the domain of α. In this paper we describe the maximal subsemigroups of the semigroup PO_n of all partial order-preserving transformations.

For $n \in I\!N$, let $X_n = \{1 < 2 < \cdots < n\}$ be a finite chain with n elements. As usual, we denote by PT_n the semigroup of all partial transformations $\alpha : X_n \to X_n$ under composition. A transformation $\alpha \in PT_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all x, y from the domain of α. As usual, PO_n denotes the subsemigroup of PT_n of all partial order-preserving transformations of X_n. This semigroup has been extensively studied. In recent years, interest in maximal subsemigroups of the transformation semigroups arises. In particular, Xiuliang Yang [6] characterized the maximal subsemigroups of the semigroup O_n of all full order-preserving transformations. Dimitrova and Koppitz [1] classified the maximal subsemigroups of the ideals of O_n. Ganyushkin and Mazorchuk [3] gave a description of the maximal subsemigroups of the semigroup POI_n of all partial order-preserving injections. In [2], Dimitrova and Koppitz characterized the maximal subsemigroups of the ideals of the semigroup POI_n. In [7], Yi constructed four types of maximal subsemigroups of the semigroup PO_n (excluding the identity map). The purpose of this paper is to give a complete classification of all maximal subsemigroups of the semigroup PO_n.

We begin by recalling some notation and definitions that are used in the paper. For the standard terms and concepts in Semigroup Theory we refer the reader to [5]. Let $\alpha \in PO_n$. We denote by dom α and im α the domain and the image of α, respectively, while $\ker\alpha := \{(x, y) \mid x, y \in \text{dom } \alpha, x\alpha = y\alpha\}$ is a convex equivalence on dom α. The natural number rank $\alpha := |\text{im } \alpha| = |\text{dom } \alpha / \ker\alpha|$ is called the rank of α. For a given subset U of PO_n, we denote by $E(U)$ its set of idempotents.

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Recall also that, for the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ on PO_n, we have

$$\begin{aligned} \alpha\mathcal{L}\beta \Longleftrightarrow \text{im}\ \alpha &= \text{im}\ \beta \\ \alpha\mathcal{R}\beta \Longleftrightarrow \ker\alpha &= \ker\beta \\ \alpha\mathcal{J}\beta \Longleftrightarrow \text{rank}\ \alpha &= \text{rank}\ \beta \\ \mathcal{H} &= \mathcal{L}\cap\mathcal{R}. \end{aligned}$$

for every transformations α and β.

The semigroup PO_n is the union of its $\mathcal{J}$-classes $J_0, J_1, J_2, \ldots, J_n$, where

$$J_k = \{\alpha \in PO_n \mid \text{rank}\ \alpha = k\}, \quad \text{for} \quad k = 0, 1, \ldots, n.$$

It follows that the ideals of the semigroup PO_n are unions of $\mathcal{J}$-classes $J_0, J_1, J_2, \ldots, J_k$, i.e. the sets

$$I_k := \{\alpha \in PO_n : \text{rank}\ \alpha \le k\}, \quad \text{with} \quad k = 0, 1, \ldots, n.$$

The $\mathcal{J}$-class J_n contains exactly one element, namely the identity, which we denote by ϵ.

We pay attention to the $\mathcal{J}$-class J_{n-1}. It is convenient to refer to an element $\alpha \in PO_n$ as belonging to the set $[k, s]$ if $|\text{dom}\ \alpha| = k$ and $|\text{im}\ \alpha| = s$ $(1 \le s \le k \le n)$. Thus the $\mathcal{J}$-class J_{n-1} is the union of $[n, n-1]$ and $[n-1, n-1]$. Let $\alpha \in [n, n-1]$ and let $\ker\alpha = \{\{1\}, \ldots, \{i-1\}, \{i, i+1\}, \{i+2\}, \ldots, \{n\}\}$ for some $i \in \{1, 2, \ldots, n-1\}$. Then, for convenience we will use the notation $\ker\alpha = (i, i+1)$.

Within $[n, n-1]$ there are exactly $(n-1)$ different $\mathcal{R}$-classes of the following form

$$R_{(i,i+1)} := \{\alpha \in J_{n-1} : \ker\alpha = (i, i+1)\}, \quad i = 1, \ldots, n-1.$$

Within $[n-1, n-1]$, which consists of one-to-one partial maps, there are exactly n different $\mathcal{R}$ - classes of the following form

$$R_i := \{\alpha \in J_{n-1} : \text{dom}\ \alpha = X_n \setminus \{i\}\}, \quad i = 1, \ldots, n.$$

The $\mathcal{L}$ - and $\mathcal{H}$-classes in J_{n-1} have the following form:

$$\begin{aligned} L_j &:= \{\alpha \in J_{n-1} : \text{im}\ \alpha = X_n \setminus \{j\}\}, \quad j = 1, \ldots, n; \\ H_{(i,i+1),j} &:= R_{(i,i+1)} \cap L_j \quad \text{and} \quad H_{i,j} := R_i \cap L_j. \end{aligned}$$

The $\mathcal{H}$-classes of PO_n are trivial, i.e. contain only one element in each case. The unique element α in the $\mathcal{H}$ - class $H_{(i,i+1),j}$ is denoted by $\alpha_{(i,i+1),j}$. Analogously, the unique transformation α in the $\mathcal{H}$ - class $H_{i,j}$ is denoted by $\alpha_{i,j}$. Since $\alpha_{(i,i+1),j}$ is an idempotent if and only if $j = i$ or $j = i+1$ and $\alpha_{i,j}$ is an idempotent if and only if $j = i$, it is easy to verify that $E(R_{(i,i+1)}) = 2$ for $i = 1, 2, \ldots, n-1$, $E(R_i) = 1$ for $i = 1, 2, \ldots, n$, $E(L_j) = 3$ for $j = 2, 3, \ldots, n-1$ and $E(L_j) = 2$ for $j = 1, n$.

Moreover, for all $\alpha, \beta \in J_{n-1}$ the product $\alpha\beta$ belongs to J_{n-1} (if and only if $\alpha\beta \in R_\alpha \cap L_\beta$) if and only if $L_\alpha \cap R_\beta$ contains an idempotent. Therefore, it is obvious that:

Lemma 1. *Let $i, k \in \{1, \ldots, n-1\}$ and $j, l, s, t \in \{1, \ldots, n\}$. Then*

$$\alpha_{(i,i+1),j}\alpha_{(k,k+1),l} = \alpha_{(i,i+1),l} \quad \textit{and} \quad \alpha_{s,j}\alpha_{(k,k+1),l} = \alpha_{s,l} \quad \Longleftrightarrow \quad j = k, k+1$$

$$\alpha_{(i,i+1),j}\alpha_{s,t} = \alpha_{(i,i+1),t} \quad \textit{and} \quad \alpha_{l,j}\alpha_{s,t} = \alpha_{l,t} \quad \Longleftrightarrow \quad j = s$$

$$L_j R_{(i,i+1)} = J_{n-1} \Longleftrightarrow j = i, i+1 \quad \textit{and} \quad L_j R_l = J_{n-1} \Longleftrightarrow j = l.$$

Let us denote by E_{n-1} the set of all idempotents of the class J_{n-1}. Further, we will use the following well known result (see [4]).

Proposition 1. *$PO_n = \langle E_{n-1}\rangle \cup \{\epsilon\} = \langle J_{n-1}\rangle \cup \{\epsilon\}$.*

Definition 1. *Let $A \subseteq X_n$ and let π be an equivalence relation on a subset Y of X_n. We say that A is a transversal of π (denoted by $A \# \pi$) if $|A \cap \bar{x}| = 1$ for every equivalence class $\bar{x} \in Y/\pi$.*

Let us denote by Λ_{n-1} the collection of all subsets of X_n of cardinality $n-1$, i.e. all sets $X_n \setminus \{i\}$ for $i = 1, 2, \ldots, n$. Let Ω_{n-1} be the collection of all convex equivalence relations on X_n with weight $n-1$, i.e. all equivalence relations $(i, i+1)$ for $i = 1, 2, \ldots, n-1$.

Remark 1. For all $\alpha \in [n, n-1]$, we have im $\alpha \in \Lambda_{n-1}$ and $\ker \alpha \in \Omega_{n-1}$. For all $\alpha \in [n-1, n-1]$, we have im $\alpha \in \Lambda_{n-1}$ and dom $\alpha \in \Lambda_{n-1}$.

Definition 2. *Let Λ be a non-empty proper subset of Λ_{n-1} and let Ω be a non-empty proper subset of Ω_{n-1}. The pair (Λ, Ω) is called a coupler of $(\Lambda_{n-1}, \Omega_{n-1})$ if the following three conditions are satisfied:*
1) *Every element of Λ is not a transversal to any element of Ω;*
2) *For every $B \in \Lambda_{n-1} \setminus \Lambda$ there exists $\pi \in \Omega$ such that $B \# \pi$;*
3) *For every $\rho \in \Omega_{n-1} \setminus \Omega$ there exists $A \in \Lambda$ such that $A \# \rho$.*

Lemma 2. *Every maximal subsemigroup of PO_n contains the ideal I_{n-2}.*

Proof. Let S be a maximal subsemigroup of PO_n. Assume that $J_{n-1} \subset S$, then $I_{n-2} \subset I_{n-1} = \langle J_{n-1}\rangle \subseteq S$. If $J_{n-1} \nsubseteq S$, then $J_{n-1} \nsubseteq \langle S \cup I_{n-2}\rangle$ since I_{n-2} is an ideal. This implies $I_{n-2} \subset S$ by the maximality of S. □

Now, we are able to present the main results of this paper, the characterization of the maximal subsemigroups of the semigroup PO_n. Recall that in [7], Yi constructed four types of maximal subsemigroups of the semigroup PO_n. They are all particular cases of the fourth type of the next theorem.

Theorem 1. *A subsemigroup S of PO_n is maximal if and only if it belongs to one of the following types:*

1. $S_\epsilon := I_{n-1}$.
2. $S_{(i,i+1)} := I_{n-2} \cup J_n \cup (J_{n-1} \setminus R_{(i,i+1)})$ *for* $i = 1, 2, \ldots, n-1$.
3. $S_i := I_{n-2} \cup J_n \cup (J_{n-1} \setminus R_i)$ *for* $i = 1, 2, \ldots, n$.
4. $S_{(\Lambda,\Omega)} := I_{n-2} \cup J_n \cup (\cup\{L_j : X_n \setminus \{j\} \in \Lambda\}) \cup$ $\cup (\cup\{R_i : X_n \setminus \{i\} \in \Lambda_{n-1} \setminus \Lambda\}) \cup (\cup\{R_{(i,i+1)} : (i, i+1) \in \Omega\})$, *where (Λ, Ω) is a coupler of $(\Lambda_{n-1}, \Omega_{n-1})$.*
5. $S_\Lambda := I_{n-2} \cup J_n \cup (\cup\{L_j : X_n \setminus \{j\} \in \Lambda\}) \cup$ $\cup (\cup\{R_i : X_n \setminus \{i\} \in \Lambda_{n-1} \setminus \Lambda\})$, *where Λ is a non-empty proper subset of Λ_{n-1} and for every $\pi \in \Omega_{n-1}$ there exists $A \in \Lambda$ such that $A \# \pi$.*

Proof. Using Lemma 1, it is not difficult to prove that each one of the given types is a subsemigroup of PO_n. Now, we are going to prove that they are maximal.

1. Since $PO_n = I_{n-1} \cup \{\epsilon\}$ and I_{n-1} is an ideal of PO_n it is clear that $S_\epsilon = I_{n-1}$ is a maximal subsemigroup of PO_n.

2. Let $\alpha \in PO_n \setminus S_{(i,i+1)}$. Then, $\alpha \in R_{(i,i+1)}$ and $\alpha \in L_j$ for some $j \in \{1, 2, \ldots, n\}$, i.e. $\alpha = \alpha_{(i,i+1),j}$. Since $L_j \cap R_j$ contains an idempotent from Lemma 1, we obtain $\alpha_{(i,i+1),j} R_j = R_{(i,i+1)}$. Therefore, since $R_j \subset S_{(i,i+1)}$, we deduce that $\langle \alpha, S_{(i,i+1)} \rangle = OP_n$, i.e. $S_{(i,i+1)}$ is maximal.

3. The proof is similar to that of $S_{(i,i+1)}$.

4. Let $\alpha \in PO_n \setminus S_{(\Lambda,\Omega)}$ and let $\alpha = \alpha_{(i,i+1),j}$. Then, $X_n \setminus \{j\} \notin \Lambda$ and so $R_j \subset S_{(\Lambda,\Omega)}$. Moreover, $(i, i+1) \notin \Omega$ and from Definition 2 it follows that $X_n \setminus \{i\} \in \Lambda$ or $X_n \setminus \{i+1\} \in \Lambda$. Without loss of generality assume that $X_n \setminus \{i\} \in \Lambda$. Then $L_i \subset S_{(\Lambda,\Omega)}$. From Lemma 1, it follows that $L_i \alpha_{(i,i+1),j} = L_j$ and $L_j R_j = J_{n-1}$. Therefore, $\langle \alpha_{(i,i+1),j}, S_{(\Lambda,\Omega)} \rangle = OP_n$. The proof when $\alpha = \alpha_{i,j}$ is similar. Hence, we deduce that $S_{(\Lambda,\Omega)}$ is maximal subsemigroup of PO_n.

5. The proof is similar to that of $S_{(\Lambda,\Omega)}$.

For the converse part let S be a maximal subsemigroup of PO_n. Then, from Lemma 2 we have $S = I_{n-2} \cup T$, where $T \subseteq J_n \cup J_{n-1} = \{\epsilon\} \cup J_{n-1}$. If $\epsilon \notin T$, then $T = J_{n-1}$ and thus $S = S_\epsilon$. If $\epsilon \in T$ then $T = \{\epsilon\} \cup T'$, where $T' \subseteq J_{n-1}$. We will consider three cases: $J_{n-1} \setminus T' \subseteq [n-1, n-1]$; $J_{n-1} \setminus T' \subseteq [n, n-1]$; $(J_{n-1} \setminus T') \cap [n-1, n-1] \neq \emptyset$ and $(J_{n-1} \setminus T') \cap [n, n-1] \neq \emptyset$.

Case 1. Let $J_{n-1} \setminus T' \subseteq [n-1, n-1]$. Since $PO_n = \langle E_{n-1} \rangle$ (see Proposition 1) there exists at least one idempotent $\alpha_{i,i} \notin S$ for some $i \in \{1, 2, \ldots, n\}$. We show that in this case $T' \cap R_i = \emptyset$. Suppose that $\alpha_{i,j} \in T'$ for some $j \in \{1, 2, \ldots, n\}$ and $j \neq i$. Then, from Lemma 1, we have $\alpha_{i,j} \alpha_{(j,j+1),i} = \alpha_{i,i} \in T'$, that is a contradiction. Therefore, we obtain $S = S_i$, by the maximality of S.

Case 2. The proof is similar to that of Case 1. Here we obtain $S = S_{(i,i+1)}$.

Case 3. Let $(J_{n-1} \setminus T') \cap [n-1, n-1] \neq \emptyset$ and $(J_{n-1} \setminus T') \cap [n, n-1] \neq \emptyset$. Since $[n, n-1] \subseteq \langle E_{n-1} \cap O_n \rangle$ it follows that there exists at least one idempotent $\alpha_{(i,i+1),i} \notin T'$ or $\alpha_{(i,i+1),i+1} \notin T'$ for some $i \in \{1, 2, \ldots, n-1\}$. Let $\alpha_{(i,i+1),i} \notin T'$ and let

$$\Lambda = \{\text{im } \alpha : \alpha \in T' \cap R_{(i,i+1)}\}.$$

Then $\Lambda \neq \emptyset$ since if $T' \cap R_{(i,i+1)} = \emptyset$ then $S \subset S_{(i,i+1)}$ which is a contradiction with the maximality of S. Moreover, $\Lambda \subset \Lambda_{n-1}$, since im $\alpha_{(i,i+1),i} \notin \Lambda$.

Now let

$$\overline{\Omega} = \{\ker \beta \in \Omega_{n-1} : \text{ there exists } \text{im } \alpha \in \Lambda \text{ such that } \text{im } \alpha \# \ker \beta\}.$$

Further, we put:

$$U = \bigcup \{\alpha_{(i,i+1),j} : X_n \setminus \{j\} \in \Lambda\},$$

$$V = \bigcup \{\alpha_{(p,p+1),q} : (p, p+1) \in \overline{\Omega} \setminus (i, i+1),\ X_n \setminus \{q\} \in \Lambda_{n-1} \setminus \Lambda\}$$

and

$$V' = \bigcup \{\alpha_{p,q} : X_n \setminus \{p\} \in \Lambda,\ X_n \setminus \{q\} \in \Lambda_{n-1} \setminus \Lambda\}.$$

Then,

$$UV = UV' = (R_{(i,i+1)} \setminus U) \cup M,$$

where $M \subseteq I_{n-2}$. Since $T' \cap (R_{(i,i+1)} \setminus U) = \emptyset$, we deduce

$$T' \cap [V \cup V' \cup (R_{(i,i+1)} \setminus U)] = \emptyset. \tag{1}$$

Finally, if $\overline{\Omega} = \Omega_{n-1}$, then by equation (1), we obtain $S \subseteq S_\Lambda$ and thus $S = S_\Lambda$ by the maximality of S. If $\overline{\Omega}$ is a proper subset of Ω_{n-1}, then we put $\Omega = \Omega_{n-1} \setminus \overline{\Omega}$. The pair (Λ, Ω) is a coupler of $(\Lambda_{n-1}, \Omega_{n-1})$ and by equation (1) we have $S \subseteq S_{(\Lambda,\Omega)}$. Therefore, we deduce $S = S_{(\Lambda,\Omega)}$ by the maximality of S. $\square$

There are exactly $2^n + 2n - 2$ maximal subsemigroups of PO_n.

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ВЪРХУ МАКСИМАЛНИТЕ ПОДПОЛУГРУПИ НА МОНОИДА ОТ ВСИЧКИ ЧАСТИЧНИ ЗАПАЗВАЩИ НАРЕДБАТА ПРЕОБРАЗОВАНИЯ

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Моноида PT_n от всички частични преобразования върху едно n-елементно множество относно операцията композиция на преобразования е изучаван в различни аспекти от редица автори. Едно частично преобразование α се нарича запазващо наредбата, ако от $x \leq y$ следва, че $x\alpha \leq y\alpha$ за всяко x, y от дефиниционното множество на α. Обект на разглеждане в настоящата работа е моноида PO_n състоящ се от всички частични запазващи наредбата преобразования. Очевидно PO_n е под-моноид на PT_n. Направена е пълна класификация на максималните подполугрупи на моноида PO_n. Доказано е, че съществуват пет различни вида максимални подполугрупи на разглеждания моноид. Броят на всички максимални подполугрупи на PO_n е точно $2^n + 2n - 2$.