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EXACT SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR ONE- AND TWO-DIMENSIONAL HEAT EQUATION*

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It is proposed an operational method for obtaining of explicit solutions of space-nonlocal BVPs for the two-dimensional heat equation. It is based on a direct three-dimensional operational calculus built on a three-dimensional convolution, combining the classical Duhamel convolution with two non-classical convolutions for the operators ∂_{xx} and ∂_{yy}. The corresponding operational calculus uses multiplier fractions instead of convolution fractions. Extensions of the Duhamel principle to the space variables are given.

1. Introduction. In M. Gutterman's paper [1] an operational calculus approach to Cauchy problems for PDEs with constant coefficients is proposed. This approach is not applicable to mixed initial-boundary value problems. According to Gutterman, such problems need new ideas and approaches. Here we use an operational calculus approach, developed in [8] to cope with BVPs for the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy} + F(x,y,t), \quad 0 < t, \quad 0 < x < a, \quad 0 < y < b, \tag{1}$$

$$u(x,y,0) = f(x,y), \qquad 0 \le x \le a, \quad 0 \le y \le b, \tag{2}$$

$$u(0,y,t) = 0, \quad \Phi_\xi\{u(\xi,y,t)\} = 0, \quad 0 \le t, \quad 0 \le y \le b, \tag{3}$$

$$u(x,0,t) = 0, \quad \Psi_\eta\{u(x,\eta,t)\} = 0, \quad 0 \le t, \quad 0 \le x \le a,$$

where Φ and Ψ are non-zero linear functionals on $C^1[0,a]$ and $C^1[0,b]$, correspondingly, $F(x,y,t)$ and $f(x,y)$ are given functions. We suppose that each of the supports $\operatorname{supp}\Phi$ and $\operatorname{supp}\Psi$ of the functionals Φ and Ψ contains at least one point, different from 0 i.e. the problem is nonlocal both with respect to x and y. In the next considerations we suppose also that Φ and Ψ satisfy the normalizing restrictions $\Phi_\xi\{\xi\} = 1$ and $\Psi_\eta\{\eta\} = 1$. These restrictions are made for the sake of simplification and they can be ousted by some unessential technical involvements.

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Key words: convolution, convolution algebra, multiplier, multiplier fractions, heat equation, nonlocal BVP, Duhamel principle, weak solution.

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2. Weak solutions of BVP (1)–(3). It is natural to look for a classical solution of the BVP (1)–(3), but, in general, the sufficient conditions for the existence of such solutions may happen to be too restrictive. That's why we introduce the notion of a *weak solution* of (1)–(3). In order to give an exact meaning of this notion, we introduce some notations. In the domain $D = [0, a] \times [0, b] \times [0, \infty)$ we consider the integral operators

$$l_t\{u(x, y, t)\} = \int_0^t u(x, y, \tau)d\tau, \tag{4}$$

and the right inverse operators L_x and L_y of $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial^2}{\partial y^2}$ given by

$$L_x\{u(x, y, t)\} = \int_0^x (x - \xi)u(\xi, y, t)d\xi - x\Phi_\xi \left\{ \int_0^\xi (\xi - \eta)u(\eta, y, t)d\eta \right\}, \tag{5}$$

and

$$L_y\{u(x, y, t)\} = \int_0^y (y - \eta)u(x, \eta, t)d\eta - y\Psi_\eta \left\{ \int_0^\eta (\eta - \varsigma)u(x, \varsigma, t)d\varsigma \right\}, \tag{6}$$

correspondingly. These operators are considered on $C(D)$. They satisfy the boundary value conditions $\Phi_x\{L_x u\} = 0$ and $\Psi_y\{L_y u\} = 0$.

Definition 1. *A function $u(x, y, t) \in C^1(D)$ is said to be a* weak solution *of problem* (1)−(3)*, if and only if it satisfies the integral relation*

$$L_x L_y u - l_t L_y u - l_t L_x u = L_x L_y f(x, y) + l_t L_x L_y F(x, y, t). \tag{7}$$

Formally, (7) is obtained from equation (1) by application of the product operator $l_t L_x L_y$, followed by using BVCs (2)–(3). It is easy to show that each classical solution of (1)–(3) is a weak solution too. If it happens that $u \in C^2(D)$, then the converse is true. Nevertheless, we can prove that each *weak solution* satisfies the BVCs (2)–(3).

Lemma 1. *Let $u \in C^1(D)$ satisfy* (7)*. Then, u satisfies BVCs* (2)−(3)*.*

Proof. Taking $t = 0$ in (7), we find $L_x L_y u(x, y, 0) = L_x L_y f(x, y)$. Hence, $u(x, y, 0) = f(x, y)$. For $x = 0$ we find $-l_t L_y u(0, y, t) = 0$ and, hence $u(0, y, t) = 0$. Next, applying Φ to (7), we get $-l_t L_y \Phi_\xi\{u(\xi, y, t)\} = 0$. If we apply $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial^2}{\partial y^2}$, then we get $\Phi_\xi\{u(\xi, y, t)\} = 0$. Analogously, we find that $u(x, 0, t) = 0$ and $\Psi_\eta\{u(x, \eta, t)\} = 0$. □

Lemma 2. *Assume that $u \in C^1(D)$ is a solution of* (7) *with continuous partial derivatives u_{xx}, u_{yy}, u_t. Then, u is a classical solution of* (1)−(3)*.*

Proof. Applying the operator $\dfrac{\partial}{\partial t}\dfrac{\partial^4}{\partial x^2 \partial y^2}$ to (7), we get $u_t = u_{xx} + u_{yy} + F(x, y, t)$. The fulfillment of the boundary value conditions follows from Lemma 1. □

Our final aim is to reduce the solution of BVP (1)–(3) to the following two nonlocal one-dimensional BVPs:

$$\begin{aligned} (8)\quad & v_t = v_{xx}, \quad 0 < t, \quad 0 < x < a, \\ & v(x, 0) = f(x), \quad 0 \le x \le a, \quad v(0, t) = 0, \quad \Phi_\xi\{v(\xi, t)\} = 0, \quad 0 \le t, \quad \text{and} \end{aligned}$$

(9) $w_t = w_{yy}, \quad 0 < t, \quad 0 < y < b,$

$$w(y,0) = g(y), \quad 0 \le y \le b, \quad w(0,t) = 0, \quad \Psi_\eta\{u(\eta,t)\} = 0, \quad 0 \le t.$$

Next, with appropriate functions $f(x)$ and $g(y)$, we consider the one-dimensional problems (8) and (9) independently of problem (1)–(3).

Definition 2. *The functions $v = v(x,t) \in C^1([0,a] \times [0,\infty))$ and $w = w(y,t) \in C^1([0,b] \times [0,\infty))$ are said to be* weak *solutions of problems* (8) *and* (9)*, if they satisfy the integral relations*

$$L_x v - l_t v = L_x f(x) \quad \text{and} \tag{10}$$

$$L_y w - l_t w = L_y g(y), \tag{11}$$

correspondingly.

Lemma 3. *If $v(x,t) \in C^1([0,a] \times [0,\infty))$ satisfies* (10)*, then $v(x,t)$ satisfies the initial and boundary value conditions $v(x,0) = f(x)$, $v(0,t) = 0$, $\Phi_\xi\{v(\xi,t)\} = 0$.*

The proof is similar to that of Lemma 1, but it is a simpler one. We skip it.

Such is the relation between problem (9) and equation (11), as well.

Lemma 4. *If $v(x,t)$ with $v_{xx}(x,t)$, $v_t(x,t) \in C([0,a] \times [0,\infty))$ satisfies* (10)*, then $v(x,t)$ is a classical solution of* (8)*.*

A similar statement holds for (11) too. The proof is similar to that of Lemma 2.

Lemma 5. *Let $v(x,t) \in C^1([0,a] \times [0,\infty))$ and $w(y,t) \in C^1([0,b] \times [0,\infty))$ be weak solutions of problems* (8) *and* (9)*, correspondingly. Then, $u(x,y,t) = v(x,t)w(y,t) \in C(D)$ is a weak solution of the BVP*

$$u_t = u_{xx} + u_{yy}, \tag{12}$$

$$u(x,y,0) = f(x)g(y), \tag{13}$$

$$u(0,y,t) = 0, \quad \Phi_\xi\{u(\xi,y,t)\} = 0, \tag{14}$$

$$u(x,0,t) = 0, \quad \Psi_\eta\{u(x,\eta,t)\} = 0.$$

in the sense of Definition 1*.*

Remark. If v and w are classical solutions, then we may assert that $u = vw$ is a classical solution of (12)–(14) too.

Proof. By Definition 2, we have:

$$L_x v = l_t v + L_x f(x), \quad L_y w = l_t w + L_y g(y). \tag{15}$$

According to Definition 1 we are to prove that:

$$L_x L_y vw - l_t L_y vw - l_t L_x vw = L_x L_y f(x)g(y).$$

Using (15), we find

$$L_x L_y vw - l_t L_y vw - l_t L_x wv = L_x v L_y w - l_t(v L_y w) - l_t(w L_x v) =$$
$$= (l_t v + L_x f(x))(l_t w + L_y g(y)) - l_t(v(l_t w + L_y g(y))) - l_t(w(l_t v + L_x f(x))) =$$
$$= (l_t v)(l_t w) - l_t(v(l_t w)) - l_t(w(l_t v)) + (L_x f(x))(L_y g(y)).$$

In order to prove the assertion of the lemma, it remains to show that $(l_t v)(l_t w) -$

$l_t(v(l_t w)) - l_t(w(l_t v)) = 0$. Indeed,

$$(l_t v)(l_t w) - l_t(v(l_t w)) - l_t(w(l_t v)) = \left(\int_0^t v(x,\tau)d\tau\right)\left(\int_0^t w(y,\tau)d\tau\right) -$$
$$- \int_0^t v(x,\tau)\left(\int_0^\tau w(y,\theta)d\theta\right)d\tau - \int_0^t w(y,\tau)\left(\int_0^\tau v(x,\theta)d\theta\right)d\tau = 0$$

Thus we proved the relation (7) for $u = vw$ and $f(x,y) = f(x)g(y)$. Hence, $u = vw$ is a weak solution of (12)–(14).

3. Convolutions. Here we briefly remind the convolutions, introduced in [8].

3.1. One-dimensional convolutions.

1) $f, g \in C[0,\infty) : (f \overset{t}{*} g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$ (Duhamel convolution).

2) $f, g \in C_x = C[0,a] : (f \overset{x}{*} g)(x) = -\frac{1}{2}\tilde{\Phi}_\xi\left\{h(x,\xi)\right\}$, (Dimovski [2])

with $\tilde{\Phi} = \Phi_\xi \circ l_\xi$ and

$$h(x,\eta) = \int_x^\eta f(\eta + x - \varsigma)g(\varsigma)d\varsigma - \int_{-x}^\eta f(|\eta - x - \varsigma|)g(|\varsigma|)\operatorname{sgn}(\varsigma(\eta - x - \varsigma))d\varsigma.$$

3) $f, g \in C_y = C[0,b] : (f \overset{y}{*} g)(y) = -\frac{1}{2}\tilde{\Psi}_\eta\left\{h(y,\eta)\right\}$, with $\tilde{\Psi} = \Psi_\eta \circ l_\eta$ (Dimovski [2]).

3.2. Two-dimensional convolutions. Define

1) $f, g \in C([0,a]\times[0,\infty)) : f(x,t) \overset{(x,t)}{*} g(x,t) = \int_0^t f(x,t-\tau) \overset{x}{*} g(x,\tau)d\tau$, (see [4]).

2) $f, g \in C([0,b]\times[0,\infty)) : f(y,t) \overset{(y,t)}{*} g(y,t) = \int_0^t f(y,t-\tau) \overset{y}{*} g(y,\tau)d\tau$, (see [**?**]).

Theorem 1. *If $f, g \in C([0,a]\times[0,b])$, then*

$$(16)\quad f(x,y) \overset{(x,y)}{*} g(x,y) = -\frac{1}{2}\tilde{\Phi}_\xi\left\{\int_x^\xi f(\xi + x - \sigma, y) \overset{y}{*} g(\sigma,y)d\sigma - \right.$$
$$\left. - \int_{-x}^\xi f(|\xi - x - \sigma|, y) \overset{y}{*} g(|\sigma|, y)\operatorname{sgn}(\xi - x - \sigma)\sigma\, d\sigma\right\}$$

is a bilinear, commutative and associative operation in $C([0,a]\times[0,b])$, such that

$$L_x L_y u(x,y,t) = \{x,y\} \overset{(x,y)}{*} u(x,y,t).$$

For a proof see [5].

Theorem 2. *Let $u, v \in C(D)$. Then, the operation*

$$(17)\qquad u(x,y,t) \overset{v}{*} (x,y,t) = \int_0^t u(x,y,t-\tau) \overset{(x,y)}{*} v(x,y,\tau)d\tau,$$

where by $\overset{x,y}{*}$ is denoted the operation (16), is a bilinear, commutative and associative operation in $C(D)$, such that

$$(18)\qquad l_t L_x L_y u(x,y,t) = \{x\,y\} * u(x,y,t).$$

For a proof see [8].

4. Ring of the multiplier fractions of $(C(D), *)$. We consider the convolution algebra $(C, *)$, where $C = C(D)$. Our direct operational calculus approach which we apply to the two-dimensional heat equation is outlined in [9]. Here we remind only some notations.

The multipliers of the form $\{u(x, y, t)\}*$ are denoted by $\{u\}$ or u and the result of the application of the operator $u*$ to a function $F \in C(D)$ is denoted simply by $\{u\}F$ or uF.

Definition 3. *Let f be a function of the variable x only in $C[0, a]$ and g be a function of the variable y only in $C[0, a]$, but both considered as functions of $C(D)$. The operators $[f]_{y,t}$ and $[g]_{x,t}$ defined by $[f]_{y,t}u = f \overset{x}{*} u$ and $[g]_{x,t}u = g \overset{y}{*} u$ are said to be partially numerical operators with respect to x, t and y, t correspondingly.*

The set of all the multipliers of the convolution algebra $(C, *)$ is a commutative ring $\mathfrak{M}$. The multiplicative set $\mathfrak{N}$ of the non-zero non-divisors of 0 in $\mathfrak{M}$ is non-empty, since at least the operators $\{x\} \overset{x}{*}$ and $\{y\} \overset{y}{*}$ are non-divisors of 0.

Next we introduce the ring $\mathfrak{M} = \mathfrak{N}^{-1}\mathfrak{M}$ of the multiplier fractions of the form $\dfrac{A}{B}$ where $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$. The standard algebraic procedure of constructing this ring, named "localization", is described, e.g. in Lang [6]. Basic for our construction are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $S_y = \dfrac{1}{L_y}$ of the multipliers L_x and L_y in $\mathfrak{M}$, correspondingly. If $u \in C^2(D)$, then, in general, $S_x u$ and $S_y u$ are different from u_{xx} and u_{yy}, but they are connected with them.

Lemma 6. *Let u_{xx}, u_{yy}, u_t be continuous on D. Then,*

$$\begin{aligned} u_{xx} &= S_x u + S_x\{(x\Phi_\xi\{1\} - 1)u(0, y, t)\} - [\Phi_\xi\{u(\xi, y, t)\}]_x, \\ u_{yy} &= S_y u + S_y\{(y\Psi_\eta\{1\} - 1)u(x, 0, t)\} - [\Psi_\eta\{u(x, \eta, t)\}]_y, \\ u_t &= su - [u(x, y, 0)]_t, \end{aligned}$$

(See [4] and [9]).

5. Formal (generalized) solution of (1)–(3). Let us consider problem (1)–(3). The equation (1) $u_t = u_{xx} + u_{yy} + F(x, y, t)$ together with the initial and boundary conditions (2) and (3) can be reduced to a single algebraic equation for u in $\mathfrak{M}$. Indeed, by Lemma 6, using (2) and (3), we get:

$$u_{xx} = S_x u, \quad u_{yy} = S_y u, \quad u_t = s\,u - [f(x, y]_t.$$

Then, (1)–(3) takes the following algebraic form in $\mathfrak{M}$:

$$(s - S_x - S_y)u = [f(x, y)]_t + F(x, y, t). \tag{19}$$

We may solve (19) in $\mathfrak{M}$, provided $s - S_x - S_y$ is a non-divisor of zero in $\mathfrak{M}$. Next, a sufficient condition for this is given by:

Theorem 3. *If $a \in \operatorname{supp}\Phi$ and $b \in \operatorname{supp}\Psi$, then the element $s - S_x - S_y$ is a non-divisor of zero in $\mathfrak{M}$.*

Remark. Theorem 3 is a special case of Theorem 13 in [9].

Corollary. *If $a \in \operatorname{supp}\Phi$ and $b \in \operatorname{supp}\Psi$, then the boundary value problem (1)–(3) has unique solution.*

Indeed, the homogeneous BVP (1)–(3) reduces to the algebraic equation $(s - S_x - S_y)u = 0$ in $\mathfrak{M}$ and, hence, $u \equiv 0$, since $s - S_x - S_y$ is a non-divisor of zero in $\mathfrak{M}$.

From now on, we suppose that $a \in \operatorname{supp} \Phi$ and $b \in \operatorname{supp} \Psi$.

The formal solution of (19) is

$$u = \frac{1}{s - S_x - S_y}([f(x, y)]_t + F(x, y, t)). \tag{20}$$

Similarly, considering the algebras $(C[0, a] \times [0, \infty), \overset{x,t}{*})$ and $(C[0, b] \times [0, \infty), \overset{y,t}{*})$ and their rings of multiplier fractions $\mathfrak{M}_{x,t}$ and $\mathfrak{M}_{y,t}$, the problem (8) and (9) have the formal solutions $v = \dfrac{1}{s - S_x}[f(x)]_t$, $w = \dfrac{1}{s - S_y}[g(y)]_t$ in $\mathfrak{M}_{x,t}$ and $\mathfrak{M}_{y,t}$ since $s - S_x$ and $s - S_y$ are non-divisors of zero (see [3]).

6. Interpretation of the formal (generalized) solution of (1)–(3) as a function.

6.1. Our next task is to interpret (20) as a function of $C([0, a] \times [0, b] \times [0, \infty))$. To this end, we consider (1)–(3) for $F(x, y, t) \equiv 0$ and $f(x, y) = xy$. We denote its weak solution, if it exists, by $U = U(x, y, t)$. We have the following algebraic representation of this solution:

$$U = \frac{1}{s - S_x - S_y}[xy]_{x,y} = \frac{1}{s - S_x - S_y}(, L_x L_y) = \frac{1}{S_x S_y (s - S_x - S_y)}.$$

Analogically, we denote the weak solutions of the problems (8) and (9) for $f(x) = x$ and $g(y) = y$ by $V = V(x, t)$ and $W = W(y, t)$, correspondingly. Then, the algebraic representations of these solutions are

$$V = \frac{1}{S_x(s - S_x)} \quad \text{and} \quad W = \frac{1}{S_y(s - S_y)}.$$

Theorem 4. *Assume that* $V = \dfrac{1}{S_x(s - S_x)}$ *and* $W = \dfrac{1}{S_y(s - S_y)}$ *are weak solutions of* (8) *and* (9) *for* $f(x) = x$ *and* $g(y) = y$*, correspondingly. Then,* $U = \dfrac{1}{S_x S_y (s - S_x - S_y)} = \{VW\}$*, where* $WV = V(x, t)W(y, t)$ *is the ordinary product of* V *and* W*, is a weak solution of* (1)–(3) *for* $F(x, y, t) \equiv 0$ *and* $f(x, y) = xy$*.*

The proof follows immediately from Lemma 5.

The generalized solution of problem (1)–(3) for arbitrary $f(x, y)$, and $F(x, y, t)$ can be represented in the form:

$$u = S_x S_y \left(\frac{1}{S_x S_y (s - S_x - S_y)}[f(x, y)]_t + \frac{1}{S_x S_y (s - S_x - S_y)} F(x, y, t) \right).$$

As a function it has the form

$$u = \frac{\partial^4}{\partial x^2 \partial y^2} \left[U \overset{x,y}{*} f(x, y) + U \overset{F}{*} (x, y, t) \right] \tag{21}$$

provided the denoted derivatives exist.

Let us consider the problem (1)–(3) for $F(x,y,t) \equiv 0$. Then,

$$u = \frac{\partial^4}{\partial x^2 \partial y^2}\left(U(x,y,t) \overset{x,y}{*} f(x,y)\right) = \frac{\partial^4}{\partial x^2 \partial y^2}\left((V(x,t)W(y,t)) \overset{x,y}{*} f(x,y)\right) =$$

$$= \frac{\partial^2}{\partial x^2}\left(V(x,t) \overset{x}{*} \frac{\partial^2}{\partial y^2}\left(W(y,t) \overset{y}{*} f(x,y)\right)\right) = V(x,t) \overset{x}{\tilde{*}} \left(W(y,t) \overset{y}{\tilde{*}} f(x,y)\right).$$

where the operations $\overset{x}{\tilde{*}}$ in $C[0,a]$ and $\overset{y}{\tilde{*}}$ in $C[0,b]$, correspondingly, are defined as

$$f(x) \overset{x}{\tilde{*}} g(x) = \frac{\partial^2}{\partial x^2}(f(x) \overset{x}{*} g(x)) \quad \text{and} \quad f(y) \overset{y}{\tilde{*}} g(y) = \frac{\partial^2}{\partial y^2}(f(y) \overset{y}{*} g(y)).$$

If $f(x,y) = f_1(x)f_2(y)$, then

$$u = (V(x,t) \overset{x}{\tilde{*}} f_1(x))(W(y,t) \overset{y}{\tilde{*}} f_2(y)).$$

This is the desired explicit solution of (1)–(3) for $f(x,y) = f_1(x)f_2(y)$.

6.2. Let us consider BVP (1)–(3) with $F(x,y,t) \equiv 0$ and

$$f(x,y) = L_x\{x\}L_y\{y\} = \frac{1}{S_x^2 S_y^2} = \left(\frac{x^3}{6} - \frac{x}{6}\Phi_\xi\{\xi^3\}\right)\left(\frac{y^3}{6} - \frac{y}{6}\Psi_\eta\{\eta^3\}\right).$$

We denote the solution of this problem by $\Omega = \Omega(x,y,t)$. Then, we have the following algebraic representation of (20):

$$\Omega = \frac{1}{s - S_x - S_y}(L_x\{x\}L_y\{y\}) = \frac{1}{S_x^2 S_y^2 (s - S_x - S_y)}.$$

Analogically, we denote the weak solutions of problems (8) and (9) for $f(x) = L_x\{x\} = \frac{1}{S_x^2} = \frac{x^3}{6} - \frac{x}{6}\Phi_\xi\{\xi^3\}$ and $g(y) = L_y\{y\} = \frac{1}{S_y^2} = \frac{y^3}{6} - \frac{y}{6}\Psi_\eta\{\eta^3\}$ by $H = H(x,t)$ and $K = K(y,t)$, correspondingly. Then, the algebraic representations of these solutions are

$$H = \frac{1}{S_x^2(s - S_x)} \quad \text{and} \quad K = \frac{1}{S_y^2(s - S_y)}.$$

Theorem 5. *Assume that* $H = \frac{1}{S_x^2(s - S_x)}$ *and* $K = \frac{1}{S_y^2(s - S_y)}$ *are weak solutions of* (8) *and* (9) *for* $f(x) = \frac{x^3}{6} - \frac{x}{6}\Phi_\xi\{\xi^3\}$ *and* $g(y) = \frac{y^3}{6} - \frac{y}{6}\Psi_\eta\{\eta^3\}$*, correspondingly. Then,* $\Omega = \frac{1}{S_x^2 S_y^2(s - S_x - S_y)} = \{HK\}$*, where* $HK = H(x,t)K(y,t)$ *is the ordinary product of* H *and* H*, is a weak solution of* (1)–(3) *for* $F(x,y,t) \equiv 0$ *and* $f(x,y) = \left(\frac{x^3}{6} - \frac{x}{6}\Phi_\xi\{\xi^3\}\right)\left(\frac{y^3}{6} - \frac{y}{6}\Psi_\eta\{\eta^3\}\right)$.

The proof follows immediately from Lemma 5.

The solution of problem (1)–(3) for arbitrary $f(x,y)$ and $F(x,y,t)$ can by represented in the form:

$$u = S_x^2 S_y^2 \left(\frac{1}{S_x^2 S_y^2(s - S_x - S_y)}[f(x,y)]_t + \frac{1}{S_x^2 S_y^2(s - S_x - S_y)}F(x,y,t)\right)$$

which can be interpreted as

$$u = \frac{\partial^8}{\partial x^4 \partial y^4}\left[\Omega \overset{x,y}{*} f(x,y) + \Omega \overset{t}{*} F(x,y,t)\right]. \tag{22}$$

Assuming some smoothness conditions for the given functions, we may assert that (22) is either weak, or classical solution of (1)–(3).

In order to reveal further the structure of the solution, we may introduce the auxiliary operations

$$f(x) \overset{x}{\circ} g(x) = \frac{\partial^2}{\partial x^2}(f(x) \overset{x}{\tilde{*}} g(x)) \quad \text{and} \quad f(y) \overset{y}{\circ} g(y) = \frac{\partial^2}{\partial y^2}(f(y) \overset{y}{\tilde{*}} g(y)).$$

Let us consider problem (1)–(3) for $F(x,y,t) \equiv 0$. We get

$$u = \frac{\partial^2}{\partial x^2}\left(V(x,t) \overset{x}{\tilde{*}} \frac{\partial^2}{\partial y^2}\left(W(y,t) \overset{y}{\tilde{*}} f(x,y)\right)\right) = V(x,t) \overset{x}{\circ} (W(y,t) \overset{y}{\circ} f(x,y)).$$

If $f(x,y) = f_1(x) f_2(y)$, then

$$u = (V(x,t) \overset{x}{\circ} f_1(x))(W(y,t) \overset{y}{\circ} f_2(y)).$$

7. Example. In the next problem, the functionals Φ and Ψ are of Samarski-Ionkin type (see [7]). Here we are looking for a classical solution of the BVP considered.

Problem. *Solve the boundary value problem:*

$$u_t = u_{xx} + u_{yy}, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \quad u(x,y,0) = f(x,y),$$

$$u(0,y,t) = 0, \quad u(x,0,t) = 0, \quad \int_0^a u(\xi,y,t)d\xi = 0, \quad \int_0^b u(x,\eta,t)d\eta = 0. \tag{23}$$

Solution. We consider the following two one-dimensional BVPs:

$$v_t = v_{xx}, \quad 0 < x < a, \quad t > 0, \quad v(x,0) = f(x), \quad v(0,t) = 0, \quad \int_0^a v(\xi,t)d\xi = 0, \tag{24}$$

and

$$w_t = w_{yy}, \quad 0 < y < b, \quad t > 0, \quad w(y,0) = g(y), \quad w(0,t) = 0, \quad \int_0^b w(\eta,t)d\eta = 0, \tag{25}$$

(here $\Phi\{f\} = \frac{2}{a^2}\int_0^a f(\xi)d\xi$ and $\Psi\{g\} = \frac{2}{b^2}\int_0^b g(\eta)d\eta$).

Let $f, g \in C[0,a]$. Then, in the case of $\Phi_\xi\{f(\xi)\} = \frac{2}{a^2}\int_0^a f(\xi)d\xi$ and $\Psi\{g\} = \frac{2}{b^2}\int_0^b g(\eta)d\eta$ the convolutions $\overset{x}{*}$ and $\overset{y}{*}$ are two times differentiable.

Lemma 7. *Let $f, g \in C[0,a]$ and $\int_0^a f(\xi)d\xi = \int_0^a g(\xi)d\xi = 0$. Then, we have*

$$(f \overset{x}{\tilde{*}} g)(x) = \frac{\partial^2}{\partial x^2}\left((f \overset{x}{*} g)(x)\right) = -\frac{1}{a^2}\left(\int_x^a f(a+x-\varsigma)g(\varsigma)d\varsigma - \right.$$
$$\left. - \int_{-x}^a f(|a-x-\varsigma|)g(|\varsigma|)\operatorname{sgn}(\varsigma(a-x-\varsigma))d\varsigma - 2\int_0^x f(x-\varsigma)g(\varsigma)d\varsigma\right).$$

Proof. By direct check. □

Lemma 8. *Let $f, g \in C^1[0,a]$ and $f(0) = g(0) = \int_0^a f(\xi)d\xi = \int_0^a g(\xi)d\xi = 0$, then*

$$f \overset{x}{\circ} g = \frac{\partial^4}{\partial x^4}\left((f \overset{x}{*} g)(x)\right) = \frac{\partial^2}{\partial x^2}\left((f \overset{x}{\tilde{*}} g)(x)\right) =$$

$$= -\frac{1}{a^2}\left(\int_x^a f'(a+x-\varsigma)g'(\varsigma)d\varsigma - \int_{-x}^a f'(|a-x-\varsigma|)\,g'(|\varsigma|)\,d\varsigma + 2\int_0^x f'(x-\varsigma)g'(\varsigma)d\varsigma\right).$$

Proof. By direct check. □

If $f(x,y) = f_1(x)f_2(y)$, then the solution of (23) is:

$$u = (V(x,t) \overset{x}{\tilde{*}} f_1(x))\,(W(y,t) \overset{y}{\tilde{*}} f_2(y)). \tag{26}$$

We are to use representation (22) from 6.2. of the solution of (20).

The solution of (24) for $f(x) = \dfrac{x^3}{6} - \dfrac{a^2x}{12} = L_x\{x\}$ is

$$H(x,t) = -2\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\left(2\left(\frac{1}{\lambda_n^3} + \frac{1}{\lambda_n^2}\lambda_n t\right)\sin\lambda_n x - \frac{1}{\lambda_n^2}x\cos\lambda_n x\right), \text{ where } \lambda_n = \frac{2n\pi}{a}.$$

Analogically, the solution of (25) for $g(y) = \dfrac{y^3}{6} - \dfrac{b^2y}{12} = L_y\{y\}$ is

$$K(y,t) = -2\sum_{m=1}^{\infty} e^{-\mu_m^2 t}\left(2\left(\frac{1}{\mu_m^3} + \frac{1}{\mu_m^2}\lambda_n t\right)\sin\mu_m y - \frac{1}{\mu_m^2}x\cos\mu_m y\right), \text{ where } \mu_m = \frac{2m\pi}{b}.$$

H and K are obtained following Ionkin's (see [7]) approach.

Theorem 6. *Let $f \in C(D)$ be such that $f_x(x,y), f_y(x,y) \in C([0,a]\times[0,b])$ and $\int_0^a f(\xi,y)d\eta = \int_0^b f(x,\eta)d\eta = 0$. Then,*

$$u = H(x,t) \overset{x}{\circ} (K(y,t) \overset{y}{\circ} f(x,y)) \tag{27}$$

is a weak solution of (23).

If suppose additionally $f(x,y) \in C^2(D)$, then (27) would be a classical solution of (23).

If $f(x,y) = f_1(x)f_2(y)$, then the solution of (23) is:

$$u = (H(x,t) \overset{x}{\circ} f_1(x))(K(y,t) \overset{y}{\circ} f_2(y)).$$

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ТОЧНИ РЕШЕНИЯ НА НЕЛОКАЛНИ ГРАНИЧНИ ЗАДАЧИ ЗА ЕДНО- И ДВУМЕРНИ УРАВНЕНИЯ НА ТОПЛОПРОВОДНОСТА

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Предложен е метод за намиране на явни решения на клас двумерни уравнения на топлопроводността с нелокални условия по пространствените променливи. Методът е основан на директно тримерно операционно смятане. Класическата дюамелова конволюция е комбинирана с две некласически конволюции за операторите ∂_{xx} и ∂_{yy} в една тримерна конволюция. Съответното операционно смятане използва мултипликаторни частни. Мултипликаторните частни позволяват да се продължи принципът на Дюамел за пространствените променливи и да се намерят явни решения на разглежданите гранични задачи. Общите разглеждания са приложени в случая на гранични условия от типа на Йонкин. Намерени са експлицитни решения в затворен вид.