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QUASI-REGULAR SOLUTIONS FOR 3D EQUATIONS OF TRICOMI AND KELDISH TYPES*

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Some 3D boundary value problems for equations of mixed type are studied. For equations of Tricomi type they are formulated by M. Protter in 1952 as three-dimensional analogues of the plane Darboux or Cauchy-Goursat problems. It is well-known that the new problems are strongly ill-posed. We formulate a new boundary value problem for equations of Keldish type and give a notion for quasi-regular solutions to this problem and to one of Protter problems. Sufficient conditions for uniqueness of such solution are found.

1. Introduction and statement of the problems. We study some 3D boundary value problems for the equation

$$L_m[u] := t^m[u_{x_1x_1} + u_{x_2x_2}] - u_{tt} + b_1u_{x_1} + b_2u_{x_2} + bu_t + cu = f, \tag{1.1}$$

where $m \in \mathbf{R}$, $m \geq 0$ and for the equation

$$\tilde{L}_m[u] := u_{x_1x_1} + u_{x_2x_2} - (t^m u_t)_t + cu = f, \tag{1.2}$$

with $m \in \mathbf{R}$, $0 < m < 2$, expressed in Cartesian coordinates (x_1, x_2, t).

Equation (1.1) for $m > 0$ is called of mixed type of the first kind (or Tricomi type) and equation (1.2) for $m > 0$, is called of mixed type of the second kind (or Keldish type).

We consider equation (1.1) in the simply connected region

$$\Omega_m := \left\{(x_1, x_2, t) :\ t > 0,\ \frac{2}{m+2}t^{\frac{m+2}{2}} < \sqrt{x_1^2 + x_2^2} < 1 - \frac{2}{m+2}t^{\frac{m+2}{2}}\right\} \subset \mathbf{R}^3,$$

bounded by the disk $\Sigma_0 := \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$ centered at the origin $O(0,0,0)$ and the characteristic surfaces of (1.1):

$$\Sigma_1^m := \left\{(x_1, x_2, t) : t > 0,\ \sqrt{x_1^2 + x_2^2} = 1 - \frac{2}{m+2}t^{\frac{m+2}{2}}\right\},$$

$$\Sigma_2^m := \left\{(x_1, x_2, t) : t > 0,\ \sqrt{x_1^2 + x_2^2} = \frac{2}{m+2}t^{\frac{m+2}{2}}\right\}.$$

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The equation (1.2) is considered in

$$\tilde{\Omega}_m := \left\{ (x_1, x_2, t) : \ t > 0, \ \frac{2}{2-m} t^{\frac{2-m}{2}} < \sqrt{x_1^2 + x_2^2} < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\} \subset \mathbf{R}^3.$$

The simply connected region $\tilde{\Omega}_m$ is bounded by the disc Σ_0 and the characteristic surfaces of (1.2):

$$\tilde{\Sigma}_1^m := \left\{ (x_1, x_2, t) : t > 0, \ \sqrt{x_1^2 + x_2^2} = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

$$\tilde{\Sigma}_2^m := \left\{ (x_1, x_2, t) : t > 0, \ \sqrt{x_1^2 + x_2^2} = \frac{2}{2-m} t^{\frac{2-m}{2}} \right\}.$$

In the case of equation of Tricomi type the parabolic plane $\{t = 0\}$ is nowhere tangential plane for the characteristic surfaces Σ_1^m and Σ_2^m of (1.1). In the case of equation of Keldish type for $0 < m < 2$ at the points of $\{t = 0\}$, the plane $\{t = 0\}$ is the tangential plane of the characteristic surfaces $\tilde{\Sigma}_1^m$ and $\tilde{\Sigma}_2^m$ of (1.2) and for $m \geq 2$ the characteristic surfaces do not intersect the plane $\{t = 0\}$.

In 1952 M. Protter formulated a new boundary value problems for equations of Triomi type, which are 3D analogues of Darboux or Cauchy–Goursat problems in the plane.

Protter's problems [16]**.** Find a solution of the equation

$$L_m^0[u] = t^m \left(u_{x_1 x_1} + u_{x_2 x_2} \right) - u_{tt} = f(x_1, x_2, t) \tag{1.3}$$

with one of the following boundary conditions

$$\begin{aligned} P1^m : &\quad u|_{\Sigma_0 \cup \Sigma_1^m} = 0, & P1^{m,*} : &\quad u|_{\Sigma_0 \cup \Sigma_2^m} = 0\,, \\ P2^m : &\quad u|_{\Sigma_1^m} = 0, u_t|_{\Sigma_0} = 0, & P2^{m,*} : &\quad u|_{\Sigma_2^m} = 0, u_t|_{\Sigma_0} = 0. \end{aligned} \tag{1.4}$$

We mention here that the boundary conditions in problem $P1^{m,*}$ (respectively $P2^{m,*}$) are the adjoint boundary conditions to problem $P1^m$ (respectively $P2^m$) for the equation (1.3) in Ω_m.

For the wave equation (i.e. $m = 0$) in 1960 P. Garabedian [5] proved uniqueness of the classical solution of Protter's problem $P1^0$ in $\mathbb{R}^4$. In contrast to the 2D case, the new problems $P1^m$ and $P2^m$ in $\mathbb{R}^3$ are not well-posed, because each of them has infinite-dimensional cokernel.

Theorem 1.1 [15]**.** *For all $m \in \mathbb{R}$, $m \geq 0$, an integer $n \geq 4$ and a_n, b_n arbitrary constants, the functions*

$$v_{n,m}(\varrho, \varphi, t) = t \varrho^{-n} \left[\varrho^2 - \left(\frac{2}{m+2} \right)^2 t^{m+2} \right]^{n-1-\frac{1}{m+2}} (a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of the homogeneous problem $P1^{m,*}$ for (1.3) *and the functions*

$$w_{n,m}(\varrho, \varphi, t) = \varrho^{-n} \left[\varrho^2 - \left(\frac{2}{m+2} \right)^2 t^{m+2} \right]^{n-1+\frac{1}{m+2}} (a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of the homogeneous problem $P2^{m,*}$ for (1.3)*.*

In order to avoid the infinitely many orthogonality conditions for classical solvability of problem $P1^m$ for equation (1.3) N. Popivanov and M. Schneider [15] introduced a new class of generalized solutions and proved that such solution exists and it is unique, but it has a very strong power-type singularity on the characteristic surface Σ_2^m.

T. Hristov and N. Popivanon [6] investigate the third boundary value Protter problem (with $u|_{\Sigma_1^m} = 0$ and $[u_t + \alpha u]|_{\Sigma_0 \backslash O} = 0, \ \alpha \in C^1(\bar{\Sigma}_0 \backslash O)$) for equation (1.1) and proved that under analogue to the classical 2D Protter's condition for lower order terms there exists unique generalized solution, which has strong power-type singularity isolated at the vertex of the inner characteristic surface Σ_1^m. In [14] they claimed the uniqueness of quasi-regular solution to the same problem, but for equation with Chaplygin operator.

Khe Kan Cher [8] has found some non-trivial solutions for the homogeneous Problems $P1^{m,*}$ and $P2^{m,*}$, but for Euler-Poison-Darboux equation.

S. A. Aldashev studied the Problems (1.4) for the equation (1.1) (see [1]), but he did not mention any possible singular solutions. In [13] A. Orshubekov claimed the existence of nontrivial solutions to the Tricomi problem for degenerating multidimensional mixed hyperbolic-parabolic type equations. About correctness of results of Aldashev and Orshubekov see our comment in [7].

Some different statements of Darboux type problems in $\mathbb{R}^3$ or some connected with them Protter problems for mixed type equations can be found in [2]–[4], [9]–[12].

In this paper we are interested in finding sufficient conditions for uniqueness of a quasi-regular solution to the following Proter problem.

Problem $P1^{m,*}$. Find a solution of (1.1) in Ω_m that satisfies the boundary conditions

$$u|_{\Sigma_0\cup\Sigma_2^m} = 0. \tag{1.5}$$

We formulate here a new boundary value Problem PK for equation of Keldish type and give notions for classical and quasi-regular solutions:

Problem PK. Find a solution of equation (1.2) that satisfies the boundary condition

$$u|_{\tilde{\Sigma}_1^m} = 0.$$

It turns out that instead of both boundary conditions given in Problems $P1^m$ and $P2^m$ on $\Sigma_0 \cup \Sigma_1^m$ for the Tricomi type equation (1.1), in the case of Keldish type equation they reduced to only one condition given on $\tilde{\Sigma}_1^m$. That means, uniqueness theorem for quasi-regular solutions of (1.2) holds if only $u|_{\tilde{\Sigma}_1^m} = 0$ is known.

2. Problem PK for equation of Keldish type.

We begin with definition of a classical solution to the new boundary value Problem PK.

Definition 2.1. *We call a function $u(x_1, x_2, t) \in C^2(\tilde{\Omega}_m)\cap C(\bar{\tilde{\Omega}}_m)$ a classical solution of Problem PK if u satisfies the equation $\tilde{L}_m[u] = f$ in $\tilde{\Omega}_m$, the boundary condition $u|_{\tilde{\Sigma}_1^m} = 0$, and $t^m u_t \to 0$, when $t \to 0$.*

The adjoint problem to the Problem PK is as follows:

Problem PK^*. Find a solution of equation (1.2) in $\tilde{\Omega}_m$ that satisfies the boundary condition

$$u|_{\tilde{\Sigma}_2^m} = 0.$$

Definition 2.2. *We call a function $u(x_1, x_2, t) \in C^2(\tilde{\Omega}_m)\cap C(\bar{\tilde{\Omega}}_m)$ a classical solution of Problem PK^* if u satisfies the equation $\tilde{L}_m[u] = f$ in $\tilde{\Omega}_m$ and the boundary condition $u|_{\tilde{\Sigma}_2^m} = 0$, and $t^m u_t \to 0$, when $t \to 0$.*

It is interesting that problem PK^* has infinite-dimensional kernel of smooth functions.

Lemma 2.3. *For all $m \in \mathbb{R}$, $0 < m < 2$, an integer $n \geq N(m)$ and a_n, b_n arbitrary* *constants, the functions*

$$g_{n,m} = \varrho^{-n}\Big[\varrho^2 - \Big(\frac{2}{2-m}\Big)^2 t^{2-m}\Big]^{n-1+\frac{1-m}{2-m}}(a_n\cos n\varphi + b_n \sin n\varphi) \tag{2.1}$$

are classical solutions of the homogeneous problem PK^* *with* $c = 0$.

Lemma 2.3 shows that for the classical solvability of the Problem PK we have infinitely many orthogonality conditions for the right-hand side function f.

Definition 2.4. *We call the function* $u(x_1, x_2, t) \in C^2(\tilde\Omega_m) \cap C(\bar{\tilde\Omega}_m)$ *a quasi-regular solution of Problem PK if*

(1) $u(x_1, x_2, t)$ *satisfies* $\tilde L_m[u] = f$ *in* $\tilde\Omega_m$, $u|_{\tilde\Sigma_1^m} = 0$;

(2) *If* $\tilde\Omega_m(\varepsilon)$ *are regions with boundaries* $\partial\tilde\Omega_m(\varepsilon)$ *lying entirely in* $\tilde\Omega_m$, *then the surface integrals along the surfaces* $\partial\tilde\Omega_m(\varepsilon)$ *which result from the application of Green's theorem to*
$$\iiint_{\tilde\Omega_m(\varepsilon)} u\tilde L_m[u]\,d\tau,\ \iiint_{\tilde\Omega_m(\varepsilon)} u_t\tilde L_m[u]\,d\tau,\ \iiint_{\tilde\Omega_m(\varepsilon)} u_{x_i}\tilde L_m[u]\,d\tau,\ i=1,2,$$
have limits when $\tilde\Omega_m(\varepsilon)$ *approaches the boundary of* $\tilde\Omega_m$ *for* $\varepsilon \to 0$.

Theorem 2.5. *The problem PK for equation* (1.2), *where* $0 < m < 2$, $c \in C^1(\bar{\tilde\Omega}_m)$, $f \in C(\bar{\tilde\Omega}_m)$ *has at most one quasi-regular solution in* $\tilde\Omega_m$ *satisfying the boundary condition* $u|_{\tilde\Sigma_1^m} = 0$ *if*

$$2c + grad(c).\vec\beta \geq 0 \ \text{ in } \ \tilde\Omega_m, \tag{2.2}$$

where

$$\vec\beta = (\beta^1, \beta^2, \beta^3) = \left(x_1, x_2, \frac{2}{2-m}t\right). \tag{2.3}$$

Remark 2.6. If $c(x_1, x_2, t) \equiv 0$ in $\tilde\Omega_m$, then the condition (2.2) is obviously fulfilled and the uniqueness Theorem 2.5 is satisfied.

3. Problem $P1^{m,*}$ for equation of Tricomi type. Here we give a definition for quasi-regular solution of the Protter problem $P1^{m,*}$ for equation (1.1) and prove the uniqueness of such solution.

Definition 3.1. *We call a function* $u(x_1, x_2, t) \in C^2(\Omega_m) \cap C(\bar\Omega_m)$ *a quasi-regular solution of problem* $P1^{m,*}$ *for equation* (1.1) *if*

(1) $u(x_1, x_2, t)$ *satisfies* $L_m[u] = f$ *in* Ω_m, $u|_{\Sigma_0\cup\Sigma_2^m} = 0$

(2) *If* $\Omega_m(\varepsilon)$ *are regions with boundaries* $\partial\Omega_m(\varepsilon)$ *lying entirely in* Ω_m, *then the integrals along* $\partial\Omega_m(\varepsilon)$ *which result from the application of Green's theorem to*
$$\iint_{\Omega_m(\varepsilon)} uL_m[u]\,d\tau,\ \iint_{\Omega_m(\varepsilon)} u_tL_m[u]\,d\tau,\ \iint_{\Omega_m(\varepsilon)} u_{x_i}L_m[u]\,d\tau,\ i=1,2,$$

have limits when $\Omega_m(\varepsilon)$ *approaches the boundary of* Ω_m *for* $\varepsilon \to 0$.

We consider (1.1) in polar coordinates (ϱ, φ, t), where $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$.

$$L_m[u] = t^m\left(\frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi}\right) - u_{tt} + a_1u_\varrho + a_2u_\varphi + bu_t + cu = f. \tag{3.1}$$

The coefficients a_1, a_2 depend on b_1, b_2 in an obvious way ($a_1 = b_1 \cos\varphi + b_2 \sin\varphi, a_2 = \varrho^{-1}(b_2 \cos\varphi - b_1 \sin\varphi)$).

Here, for the sake of simplicity, we assume that all the coefficients of (3.1) depend only on ϱ and t.

Theorem 3.2. *The Problem $P1^{m,*}$ for equation*

$$L_m[u] = \sum_{n=0}^{N} \left\{ f_n^{(1)}(\varrho, t) \cos n\varphi + f_n^{(2)}(\varrho, t) \sin n\varphi \right\}, \tag{3.2}$$

where $m \geq 0$, $m \in \mathbb{R}$, $N \in \mathbb{N} \cup \{0\}$, $a_1, b \in C^2(\Omega_m)$, $c \in C^1(\Omega_m)$ has at most one quasi-regular solution

$$u(\varrho, \varphi, t) = \sum_{n=0}^{N} \left\{ u_n^{(1)}(\varrho, t) \cos n\varphi + u_n^{(2)}(\varrho, t) \sin n\varphi \right\}$$

in Ω_m, if $a_2 \equiv 0$ and for all $0 \leq n \leq N$

$$a_{00} > 0, \quad mt^{\frac{m-2}{2}} + a_1 + t^m \varrho^{-1} + t^{\frac{m}{2}} b \geq 0 \quad in \quad \Omega_m.$$

Hereby

$$\alpha^1 = -\mu t^m, \quad \alpha^2 = -\mu t^{\frac{m}{2}}, \quad 2\alpha^0 = -\mu \left\{ \frac{m}{2} t^{\frac{m-2}{2}} + a_1 + t^m \varrho^{-1} - t^{\frac{m}{2}} b \right\},$$

$$\mu = 1 - \varrho - \frac{2}{m+2} t^{\frac{m+2}{2}}, R = c(\varrho, t) - \frac{n^2}{\varrho^2} t^m$$

$$a_{00} = -2\alpha^0 R + (R\alpha^1)_\varrho + (R\alpha^2)_t - t^m \alpha^0_{\varrho\varrho} + \alpha^0_{tt} + (\alpha^0(a_1 + t^m \varrho^{-1}))_\varrho + (\alpha^0 b)_t.$$

Example 3.3. *Consider the equation*

$$\begin{aligned} L_0[u] &= \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} + b(\varrho, t)(u_\varrho + u_t) + \frac{k}{\varrho^2} u \\ &= \sum_{n=0}^{N} \left\{ f_n^{(1)}(\varrho, t) \cos n\varphi + f_n^{(2)}(\varrho, t) \sin n\varphi \right\} \end{aligned} \tag{3.3}$$

in Ω_m, where k is a constant and $N > 0$.

If

$$b(\varrho, t) \geq 0, \quad b_\varrho + b_t \leq 0, \quad k \geq N^2, \tag{3.4}$$

then Theorem 3.2 gives that the problem $P1^{0,*}$ for equation (3.3) has at most one quasi-regular solution

$$u(\varrho, \varphi, t) = \sum_{n=0}^{N} \left\{ u_n^{(1)}(\varrho, t) \cos n\varphi + u_n^{(2)}(\varrho, t) \sin n\varphi \right\}.$$

Remark 3.4. For $k = 0$, $b(\varrho, t) = 0$, $N > 1$ in (3.3) the condition $a_{00} > 0$ in Ω_0 does not hold and thus the uniqueness Theorem 3.2 is not valid. But this is already known, since in this case infinite many of classical solutions of problem $P1^{0,*}$ exist (see Theorem 1.1).

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КВАЗИРЕГУЛЯРНИ РЕШЕНИЯ НА ТРИМЕРНИ УРАВНЕНИЯ ОТ ТИПА НА ТРИКОМИ И ТИПА НА КЕЛДИШ

Цветан Д. Христов, Недю Ив. Попиванов, Манфред Шнайдер

Изучени са някои тримерни гранични задачи за уравнения от смесен тип. За уравнения от типа на Трикоми те са формулирани от М. Протер през 1952, като тримерни аналози на задачите на Дарбу или Коши–Гурса в равнината. Добре известно е, че новите задачи са некоректни. Ние формулираме нова гранична задача за уравнения от типа на Келдиш и даваме понятие за квазиругулярно решение на тази задача и на една от задачите на Протер. Намерени са достатъчни условия за единственост на такива решения.