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ON THE OPTIMAL ALLOCATIONS IN ECONOMY WITH
FIXED TOTAL RESOURCES*

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In this paper we consider a mathematical model of economy with fixed total resources when the numbers of agents and goods are finite. We discuss the role of some assumptions for preference relation of the economical agents that affect the characteristics of the optimal allocations. It is proved that a set of optimal allocations is contractible and has the fixed point property.

1. Introduction. In mathematical economics a concept of optimal allocations (or Pareto-optimal allocations) is related to a concept of general equilibrium. It is known that every equilibrium allocation is optimal. This statement is the first fundamental theorem of welfare economics. In the present paper we examine some characteristics of the set of optimal allocations (the Pareto-optimal set) not using the fact of equilibrium [1, 6, 7].

Consider a mathematical model of economy with fixed total resources when the numbers of agents and goods are finite. This economy is defined by a set A of economical agents, $|A| = n \geq 2$, $J_A = \{1, 2, \dots, n\}$, a set G of perfectly divisible goods, $|G| = m \geq 2$, $J_G = \{1, 2, \dots, m\}$, and a vector $r \in R_{++}^m$ of fixed total resources. We denote by $\Sigma = \{x(x^1, x^2, \dots, x^n) \in R_+^{mn} \mid \sum_{i=1}^n x^i = r\}$ the space of individual allocations, where agent $a_i \in A$ owns of $x^i(x_1^i, x_2^i, \dots, x_m^i) \in R_+^m$, a number $x_j^i \geq 0$ shows the quantity of $g_j \in G$ property of this agent. Let each agent $a_i \in A$ have a binary weak preference relation \succeq_i defined on $X = [0, r]$, $\succeq_i \subset X \times X$. A strict preference relation \succ_i is associated with \succeq_i as usually: $y \succ_i x$ is equivalent to $y \succeq_i x$ and not $x \succeq_i y$ for $x, y \in X$. In our model, let d be the Euclidean metric in R^{mn} and τ be the topology induced by d .

Let every relation \succeq_i of $\{\succeq_i\}_{i=1}^n$ be reflexive, transitive, complete and continuous on X . Thus each binary relation \succeq_i can be represented by a continuous utility function $u_i : X \rightarrow R$ such that for every $x, y \in X$, $x \succeq_i y$ is equivalent to $u_i(x) \geq u_i(y)$ [2, Theorem 5.1] [4, Theorem 9]. The preference relation (or the utility function) of the economical agents is a measure for the status quo of the agents.

Now, we are ready to introduce the definition of the Pareto-optimal allocations.

Definition 1. (a) An allocation $x \in \Sigma$ is called a strictly Pareto-optimal allocation if and only if there does not exist $y \in \Sigma$ such that $y^i \succeq_i x^i$ (or $u_i(y^i) \geq u_i(x^i)$) for all $i \in J_A$ and $x \neq y$. The set of the strictly Pareto-optimal allocations on Σ is denoted by

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SP and it is called strictly Pareto-optimal set.

(b) An allocation $x \in \Sigma$ is called Pareto-optimal allocation if and only if there does not exist $y \in \Sigma$ such that $y^i \succeq_i x^i$ (or $u_i(y^i) \geq u_i(x^i)$) for all $i \in J_A$ and $y^k \succ_k x^k$ (or $u_k(y^k) > u_k(x^k)$) for some $k \in J_A$. The set of the Pareto-optimal allocations on Σ is denoted by P and it is called Pareto-optimal set.

(c) An allocation $x \in \Sigma$ is called weakly Pareto-optimal allocation if and only if there does not exist $y \in \Sigma$ such that $y^i \succ_i x^i$ (or $u_i(y^i) > u_i(x^i)$) for all $i \in J_A$. The set of the weakly Pareto-optimal allocations on Σ is denoted by WP and it is called weakly Pareto-optimal set.

It is well-known that $SP \subset P \subset WP$, P is nonempty and SP can be empty [1, 5].

From the definition it can be seen that the allocations of the Pareto-optimal sets are defined only by the preference relations $\{\succeq\}_{i=1}^n$ (or the utility functions $\{u_i\}_{i=1}^n$) of the agents. The Pareto-optimal allocations not related to the prices system of the economy and the budgetary limitations of the economical agents. It can also be shown that the Pareto-optimal allocations depend only on the preference relations $\{\succeq\}_{i=1}^n$ and not depend on the choice of utility functions $\{u_i\}_{i=1}^n$.

Remark 1. Let X and Y be topological spaces. A homotopy between two continuous functions $f, g : X \rightarrow Y$ is defined to be a continuous function $H : X \times [0; 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Note that we can consider the homotopy H as a continuous deformation of f to g ?

Definition 2. (a) The set $Y \subset X$ is a retract of X if and only if there exists a continuous function $r : X \rightarrow Y$ such that $r(x) = x$ for all $x \in Y$. The function r is called retraction.

(b) The set $Y \subset X$ is a deformation retract of X if and only if there exist a retraction $r : X \rightarrow Y$ and a homotopy $H : X \times [0; 1] \rightarrow X$ such that $H(x, 0) = x$ and $H(x, 1) = r(x)$ for all $x \in X$.

(c) The set Y is contractible (contractible to a point) if and only if there exists a point $a \in Y$ such that $\{a\}$ is a deformation retract of Y .

Definition 3. The topological space Y is said to have the fixed point property if and only if every continuous mapping $h : Y \rightarrow Y$ of this space into itself has a fixed point, i.e. there is a point $x \in Y$ such that $x = h(x)$.

Remark 2. It is known that convexity implies contractibility and fixed point property, but the converse does not hold in general. Contractibility and fixed point property of sets are preserved under retraction. This means that: (i) if X is contractible and Y is a retract of X , then Y is contractible too. (ii) if X has the fixed point property and Y is a retract of X , then Y has the fixed point property too. It is easy to verify that the space Σ is convex and compact; therefore, it is contractible and has the fixed point property.

We introduce the following notations: for every two vectors $x, y \in R^m$, $x(x_1, x_2, \dots, x_m) \geq y(y_1, y_2, \dots, y_m)$ means $x_j \geq y_j$ for all $j \in J_G$ and $x(x_1, x_2, \dots, x_m) > y(y_1, y_2, \dots, y_m)$ means $x_j > y_j$ for all $j \in J_G$.

The weak preference relation \succeq_i is called monotone on X if and only if for every $x, y \in X$ such that $x \neq y$ and $x \geq y$ imply that $x \succ_i y$ (or $u_i(x) > u_i(y)$). In this case, the utility function u_i is monotone on X .

The weak preference relation \succeq_i is called convex on X if and only if for every $x, y \in X$ such that $y \succ_i x$ and $t \in (0; 1]$ imply $ty + (1 - t)x \succ_i x$. For utility function u_i we find that for every $x, y \in X$ such that $x \neq y$ and $t \in (0; 1)$ imply that $u_i(tx + (1 - t)y) > \min(u_i(x), u_i(y))$. In the other words, the utility function u_i is strictly quasi-concave on X .

Let define multifunction $\rho : \Sigma \Rightarrow \Sigma$ such that $\rho(x) = \{y \in \Sigma | u_i(y^i) \geq u_i(x^i), i \in J_A\}$ for $x \in \Sigma$. We can easily check that: (i) ρ is compact-valued; (ii) if the utility functions $\{u_i\}_{i=1}^n$ are quasi-concave, then ρ is convex-valued.

2. Main result. In this section, we discuss the role of the following assumptions that affect the characteristics of the Pareto-optimal sets.

Assumption 1. If $\{x_i\}_{i=0}^\infty \subset \Sigma$ and $\lim_{k \rightarrow \infty} d(x_k, x_0) = 0$, then $\lim_{k \rightarrow \infty} d(y_0, \rho(x_k)) = 0$ for all $y_0 \in \rho(x_0)$.

Assumption 2. The weak preference relations $\{\succeq_i\}_{i=1}^n$ are monotone on X .

Assumption 3. The weak preference relations $\{\succeq_i\}_{i=1}^n$ are convex on X .

Remark 3. In [5], it is proved that if the utility functions are strictly quasi-concave, then $WP = P = SP$.

Remark 4. In [6], it is proved that a Pareto-optimal set is path-wise connected when the utility functions are monotone, concave and strictly quasi-concave.

Theorem 1. If the Assumptions 1, 2 and 3 hold, then P is contractible and has the fixed point property.

In order to give the proof of Theorem 1, we will prove some lemmas.

Lemma 1. If the Assumption 1 holds, then ρ is continuous on Σ .

Proof. First, we prove that if $\{x_k\}_{k=1}^\infty \subset \Sigma$ and $\{y_k\}_{k=1}^\infty \subset \Sigma$ are a pair of sequences such that $\lim_{k \rightarrow \infty} x_k = x_0 \in \Sigma$ and $y_k \in \rho(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^\infty$ whose limit belongs to $\rho(x_0)$.

The assumption $y_k \in \rho(x_k)$ for all $k \in N$ implies that $u_i(y_k^i) \geq u_i(x_k^i)$ for all $k \in N$ and for all $i \in J_A$. From the condition $\{y_k\}_{k=1}^\infty \subset \Sigma$ it follows that there exists a convergent sequence $\{\bar{y}_k\}_{k=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} \bar{y}_k = y_0 \in \Sigma$, $\{\bar{x}_k\}_{k=1}^\infty \subset \{x_k\}_{k=1}^\infty$, $\lim_{k \rightarrow \infty} \bar{x}_k = x_0$ and $\bar{y}_k \in \rho(\bar{x}_k)$. So, we can deduce that $u_i(\bar{y}_k^i) \geq u_i(\bar{x}_k^i)$ for all $k \in N$ and all $i \in J_A$. Taking the limit as $k \rightarrow \infty$, we get $u_i(y_0^i) \geq u_i(x_0^i)$ for all $i \in J_A$, i.e. $y_0 \in \rho(x_0)$.

This means that ρ is upper semi-continuous on Σ .

From the Assumption 1 it follows that ρ is lower semi-continuous on Σ , see also [3]. Finally, we obtain that the multifunction ρ is continuous on Σ . The lemma is proved. \square

Lemma 2. If $x \in \Sigma$ and the Assumption 3 holds, then $x \in P$ is equivalent to $|\rho(x)|=1$.

Proof. Let $x \in P$ and assume that $|\rho(x)| > 1$. From both conditions $x \in \rho(x)$ and $\{x\} \neq \rho(x)$ it follows that there exists $y \in \rho(x) \setminus \{x\}$ such that $u_i(y^i) \geq u_i(x^i)$ for all $i \in J_A$.

Let $t \in (0; 1)$ and $z = tx + (1 - t)y$, then, $z \in \rho(x)$. Fix a number $i \in J_A$. There are two cases as follows: (i) if $x^i = y^i$, then $u_i(z^i) = u_i(y^i) = u_i(x^i)$; (ii) if $x^i \neq y^i$, then $u_i(z^i) > \min(u_i(y^i), u_i(x^i)) = u_i(x^i)$.

Of course, $x \neq y$ implies that $x^k \neq y^k$ for some $k \in J_A$. This allows us to conclude that $u_i(z^i) \geq u_i(x^i)$ for all $i \in J_A$ and $u_k(z^k) > u_k(x^k)$ for some $k \in J_A$, which contradicts the assumption $x \in P$. Therefore, we obtain $|\rho(x)| = 1$.

Conversely, let $|\rho(x)| = 1$ and assume that $x \notin P$. Then, from $x \notin P$ it follows that there exists $y \in \Sigma$ such that $u_i(y^i) \geq u_i(x^i)$ for all $i \in J_A$ and $u_k(y^k) > u_k(x^k)$ for some $k \in J_A$. Obviously, we have that $y \in \rho(x)$ and $x \neq y$, which contradicts the assumption $|\rho(x)| = 1$. Therefore, we obtain that $x \in P$. The lemma is proved. \square

Choose $a_1 \in A$ and $x \in \Sigma$. Now we consider the optimization problem: maximize $u_1(y^1)$, subject to $y \in \rho(x)$. By letting x to vary over all of Σ we can identify different optimal allocation. This optimization technique allow us to find the whole Pareto-optimal set, see also [7]. Thus we proved that: (i) for each $x \in \Sigma$ this optimization problem has a unique solution $\bar{x} \in P$; (ii) the function $x \in \Sigma \mapsto \bar{x} \in P$ is continuous.

This note allows us to formulate the following lemma.

Lemma 3. *If $x \in \Sigma$ and the Assumptions 2 and 3 hold, then $|\text{Arg max}(u_1, \rho(x))| = 1$.*

Proof. Clearly, $|\text{Arg max}(u_1, \rho(x))| \geq 1$. Let assume that $|\text{Arg max}(u_1, \rho(x))| > 1$ and $y_1, y_2 \in \Sigma$, $y_1 \neq y_2$, $y_1^1, y_2^1 \in \text{Arg max}(u_1, \rho(x))$, $t \in (0; 1)$ and $z = ty_1 + (1-t)y_2$. It is easy to see that the set $\text{Arg max}(u_1, \rho(x))$ is convex and $y_1^1 \neq y_2^1$; therefore, $z \in \text{Arg max}(f, \rho(x))$ and $u_1(z^1) = u_1(y_1^1) = u_1(y_2^1)$.

From $y_1 \neq y_2$ and $y_1^1 \neq y_2^1$ it follows that there exists $k \in J_A \setminus \{1\}$ such that $y_1^k \neq y_2^k$. As a result we obtain that $u_k(z^k) > \min(u_k(y_1^k), u_k(y_2^k))$. In this case there exists $y \in \rho(x)$ such that $u_k(z^k) > u_k(y^k) > \min(u_k(y_1^k), u_k(y_2^k))$, $u_1(y^1) > u_1(z^1)$ and $y^j = z^j$ for all $j \in J_A \setminus \{1, k\}$ (see Assumption 2). Thus we get that $u_1(y^1) > u_1(z^1)$ and $y, z \in \rho(x)$, but $z \in \text{Arg max}(f, \rho(x))$. This leads to a contradiction. Finally, we obtain that $|\text{Arg max}(u_1, \rho(x))| = 1$. The lemma is proved. \square

Lemma 4. *If $x \in \Sigma$ and the Assumptions 2 and 3 hold, then $\text{Arg max}(u_1, \rho(x)) \subset P$.*

Proof. Let $y \in \Sigma$, $y^1 \in \text{Arg max}(u_1, \rho(x))$ and assume that $y \notin P$. From the assumption $y \notin P$ it follows that there exists $z \in \Sigma$ such that $u_i(z^i) \geq u_i(y^i)$ for all $i \in J_A$ and $u_k(z^k) > u_k(y^k)$ for some $k \in J_A$. As a result we obtain that $z \in \rho(x)$.

On one hand, the condition $u_i(z^i) \geq u_i(y^i)$ for all $i \in J_A$ implies that $z \in \text{Arg max}(u_1, \rho(x))$.

On the other hand, the condition $u_k(z^k) > u_k(y^k)$ for some $k \in J_A$ implies that $z \neq y$.

But, in Lemma 3 we have proved that $|\text{Arg max}(u_1, \rho(x))| = 1$. This leads to a contradiction, and therefore, we obtain $y \in P$. The lemma is proved. \square

Using the results of Lemmas 3 and 4 we are in position to construct a function $r : \Sigma \rightarrow P$ such that $r(x) \in \text{Arg max}(u_1, \rho(x))$ for all $x \in \Sigma$.

Now, our attention is focused on the function r .

Lemma 5. *$r(P) = P$ and $r(\Sigma) = P$.*

Proof. Applying now Lemmas 2, 3 and 4, we get that $r(P) = P$. According to Lemma 4, from the fact that $P \subset \Sigma$, we obtain $r(\Sigma) = P$. The lemma is proved. \square

Remark 5. In the proof of Theorem 1 we use Maximum Theorem [3, Maximum Theorem] [4, Theorem 6.5]. Let X be a topological space. If $F : X \rightarrow R$ is a continuous function and $B : X \Rightarrow X$ is a continuous compact-valued multifunction, then the multifunction $\gamma : X \Rightarrow X$ defined by $\gamma(y) = \{x \in B(y) | F(x) \geq F(x'), x' \in B(y)\}$ is upper semi-continuous and compact-valued, and the function $f : X \rightarrow R$ defined by $f(y) = F(\gamma(y))$ is continuous.

Lemma 6. r is continuous on Σ .

Proof. By the previous remark, let $X = \Sigma$, $F(x) = u_1(x^1)$ for all $x(x^1, x^2, \dots, x^n) \in \Sigma$ and $B = \rho$. Applying now Lemma 3 we derive $|\gamma(y)| = 1$ for all $y \in \Sigma$. This means that γ is function and $\gamma = r$. It is known that every upper semi-continuous point-to-point multifunction is continuous, i.e. r is continuous on Σ . The lemma is proved. \square

Proof of Theorem 1. According to Lemmas 5 and 6 we obtain that r is a retraction of Σ to P . From Remark 2 it follows that P is contractible and has the fixed point property. The theorem is proved. \square

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ВЪРХУ ОПТИМАЛНО РАЗПРЕДЕЛЕНИТЕ ДЯЛОВЕ В ИКОНОМИКА С ФИКСИРАНИ ОБЩИ РЕСУРСИ

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В тази статия се разглежда математически модел на икономика с фиксирани общи ресурси, както и краен брой агенти и блага. Обсъжда се ролята на някои предположения за отношенията на предпочитание на икономическите агенти, които влияят на характеристиките на оптимално разпределените дялове. Доказва се, че множеството на оптимално разпределените дялове е свиваемо и притежава свойството на неподвижната точка.