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δ_K-SMALL SETS IN GRAPHS*

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Let G be a simple n-vertex graph and $W \subseteq \mathrm{V}(G)$. We say that W is a δ_k-small set if

$$\sqrt[k]{\frac{\sum_{v\in W} d^k(v)}{|W|}} \leq n - |W|\,.$$

Let $\varphi^{(k)}(G)$ denote the smallest natural number r such that $\mathrm{V}(G)$ decomposes into r δ_k-small sets, and let $\alpha^{(k)}(G)$ denote the maximal number of vertices in a δ_k-small set of G. In this paper we obtain bounds for $\alpha^{(k)}(G)$ and $\varphi^{(k)}(G)$. Since $\varphi^{(k)}(G) \leq \omega(G) \leq \chi(G)$ and $\alpha(G) \leq \alpha^{(k)}(G)$, we obtain also bounds for the clique number $\omega(G)$, the chromatic number $\chi(G)$ and the independence number $\alpha(G)$.

1. Introduction. We consider only finite, non-oriented graphs without loops and multiple edges. We shall use the following notations:

$\mathrm{V}(G)$ – the vertex set of G;
$e(G)$ – the number of edges of G;
$\omega(G)$ – the clique number of G;
$\alpha(G)$ – the independence number of G;
$\chi(G)$ – the chromatic number of G;
$d(v)$ – the degree of a vertex v;
$\Delta(G)$ – the maximal degree of G;
$\delta(G)$ – the minimal degree of G.

All undefined notation are from [8].

Definition 1. Let G be an n-vertex graph and $W \subseteq \mathrm{V}(G)$. We say that W is a *small set* in the graph G if

$$d(v) \leq n - |W|\,, \text{ for all } v \in W.$$

With $\varphi(G)$ we denote the smallest natural number r such that $\mathrm{V}(G)$ decomposes into r small sets.

The number $\varphi(G)$ is defined for the first time in [6]. Some properties of $\varphi(G)$ are proved in [6] and [2]. Further $\varphi(G)$ is more thoroughly investigated in [1]. There an effective algorithm for the calculation of $\varphi(G)$ is given. First of all let us note the following bounds for $\varphi(G)$.

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Proposition 1.1 ([1]).

$$\left\lceil \frac{n}{n - \mathrm{d}_1(G)} \right\rceil \le \varphi(G) \le \left\lceil \frac{n}{n - \Delta(G)} \right\rceil ,$$

where $\mathrm{d}_1(G)$ *is the average degree of the graph* G.

Let G be a graph and $W \subseteq \mathrm{V}(G)$. We define

$$\mathrm{d}_k(W) = \sqrt[k]{\frac{\sum\limits_{v \in W} d^k(v)}{|W|}}, \qquad \mathrm{d}_k(G) = \mathrm{d}_k\big(\mathrm{V}(G)\big).$$

Definition 2. Let G be an n-vertex graph and $W \subseteq \mathrm{V}(G)$. We say that W is a δ_k*-small set* of G if

$$\mathrm{d}_k(W) \le n - |W| .$$

With $\varphi^{(k)}(G)$ we denote the minimal number of δ_k-sets of G into which $\mathrm{V}(G)$ decomposes.

Remark 1. δ_1-small sets are defined in [1] as β-small sets and $\varphi^{(1)}(G)$ is denoted by $\varphi^{\beta}(G)$. Also in [1] it is proven

Proposition 1.2 ([1]).

$$\varphi^{(1)}(G) \ge \left\lceil \frac{n}{n - \mathrm{d}_1(G)} \right\rceil .$$

Further we shall need the following

Proposition 1.3. *Let* G *be an* n*-vertex graph. Then*

(i) *Every small set of* G *is a* δ_k*-small set of* G *for all natural* k.

(ii) *Every* δ_{k-1}*-small set of* G *is a* δ_k*-small set of* G.

Proof. Let W be a small set of G. Then $d(v) \le n - |W|$, $\forall v \in W$. Therefore $\mathrm{d}_k(W) \le n - |W|$, i. e. W is a δ_k-small set.

The statement in (ii) follows from the inequality $\mathrm{d}_{k-1}(W) \le \mathrm{d}_k(W)$ (cf. [4, 5]). □

Let us note that if G is an r-regular graph then $\mathrm{d}_k(W) = r$ for all natural k. So, in this case, every δ_k-set of G is a small set of G.

In this paper we shall prove that for a given graph G and for sufficiently large natural k every δ_k-small set of G is a small set of G (Theorem 2.1).

Proposition 1.4. *Let* G *be a graph. Then*

$$\varphi^{(1)}(G) \le \varphi^{(2)}(G) \le \cdots \le \varphi^{(k)}(G) \le \cdots \le \varphi(G) \le \omega(G) \le \chi(G).$$

Proof. The inequality $\chi(G) \ge \omega(G)$ is obvious. The inequality $\varphi(G) \le \omega(G)$ is proven in [6] (see also [1]). The inequality $\varphi^{(k)}(G) \le \varphi(G)$ follows from Proposition 1.3 (i) and the inequlity $\varphi^{(k-1)}(G) \le \varphi^{(k)}(G)$ follows from Proposition 1.3 (ii). □

According to Proposition 1.4 every lower bound for $\varphi^{(k)}(G)$ is a lower bound for $\varphi(G)$, $\omega(G)$ and $\chi(G)$. In this paper we shall obtain a lower bound for $\varphi^{(k)}(G)$ (Theorem 3.2) from which we shall derive new lower bounds for $\varphi(G)$, $\omega(G)$ and $\chi(G)$. As a corollary we shall get and some results for $\varphi(G)$, $\omega(G)$ and $\chi(G)$ already from [1] and [2].

Proposition 1.5.

$$\left\lceil \frac{n}{n - \mathrm{d}_1(G)} \right\rceil \le \varphi^{(k)}(G) \le \left\lceil \frac{n}{n - \Delta(G)} \right\rceil .$$

Proof. The right inequality follows from Proposition 1.1 and Proposition 1.4. The left inequality follows from Proposition 1.2 and Proposition 1.4. □

2. Strengthening Proposition 1.4.

Theorem 2.1. *Let G be a graph. There exists a natural $k_0 = k_0(G)$ such that for all $k \geq k_0$ we have*

(i) *Every δ_k-small set of G is a small set of G.*

(ii) $\varphi^{(1)}(G) \leq \cdots \leq \varphi^{(k_0)}(G) = \varphi^{(k_0+1)}(G) = \cdots = \varphi(G)$.

Proof. Fix a subset of $\mathrm{V}(G)$, say W, and let $\Delta(W) = \max\{d(v) \mid v \in W\}$. Then $\mathrm{d}_k(W) \leq \Delta(W)$ and $\lim\limits_{k\to\infty} \mathrm{d}_k(W) = \Delta(W)$ (see [4]).

Therefore, since $\mathrm{V}(G)$ has only finitely many subsets, there exists k_0 such that for arbitrary $W \subseteq \mathrm{V}(G)$

$$\Delta(W) - \frac{1}{2} \leq \mathrm{d}_k(W), \text{ if } k \geq k_0. \tag{2.1}$$

Let us suppose now that W is a δ_k-small set of G and $k \geq k_0$, i. e.

$$\mathrm{d}_k(W) \leq n - |W|\,. \tag{2.2}$$

From (2.1) and (2.2) we have that

$$\Delta(W) - \frac{1}{2} \leq n - |W|\,.$$

Since $\Delta(W)$ and $n - |W|$ are integers, from the last inequality we derive that $\Delta(W) \leq n - |W|$. From the definition of $\Delta(W)$ it follows $d(v) \leq n - |W|$ for all $v \in W$, i. e. W is a small set. Thereby (i) is proven. The statement (ii) obviously follows from (i). □

3. Lower bounds for $\mathbf{d}_k(G)$ and $\varphi^{(k)}(G)$.

Lemma 3.1. *Let $\beta_1, \beta_2, \ldots, \beta_r \in [0,1]$ and $\beta_1 + \beta_2 + \cdots + \beta_r = r - 1$. Then for all positive integer $k \leq r$*

$$\sum_{i=1}^{r} (1 - \beta_i)\beta_i^k \leq \left(\frac{r-1}{r}\right)^k. \tag{3.1}$$

Proof. The case $k = r$ is proven in [1]. That is why we suppose that $k \leq r - 1$. For all natural n we define

$$S_n = \beta_1^n + \beta_2^n + \cdots + \beta_r^n.$$

We can rewrite the inequality (3.1) in following way

$$S_k - S_{k+1} \leq \left(\frac{r-1}{r}\right)^k. \tag{3.2}$$

Since

$$\frac{r-1}{r} = \frac{S_1}{r} \leq \sqrt[k]{\frac{S_k}{r}} \leq \sqrt[k+1]{\frac{S_{k+1}}{r}} \quad \text{(cf. [4, 5])},$$

we have

$$S_{k+1} \geq \frac{1}{\sqrt[k]{r}} S_k^{\frac{k+1}{k}} \tag{3.3}$$

and

$$S_k \geq \frac{(r-1)^k}{r^{k-1}}. \tag{3.4}$$

From (3.3) we see that

$$S_k - S_{k+1} \le S_k - \frac{1}{\sqrt[k]{r}} S_k^{\frac{k+1}{k}}. \tag{3.5}$$

We consider the function

$$f(x) = x - \frac{1}{\sqrt[k]{r}} x^{\frac{k+1}{k}}, \quad x > 0.$$

According to (3.2) and (3.5) it is sufficient to prove that

$$f(S_k) \le \left(\frac{r-1}{r}\right)^k.$$

From $f'(x) = 1 - \dfrac{k+1}{k\sqrt[k]{r}} x^{\frac{1}{k}}$, it follows that $f'(x)$ has unique positive root

$$x_0 = \frac{rk^k}{(k+1)^k}$$

and $f(x)$ decreases in $[x_0, \infty)$. According to (3.4), $S_k \ge \dfrac{(r-1)^k}{r^{k-1}}$. Since $k \le r-1$, $\dfrac{(r-1)^k}{r^{k-1}} \ge x_0$. Therefore

$$f(S_k) \le f\left(\frac{(r-1)^k}{r^{k-1}}\right) = \left(\frac{r-1}{r}\right)^k.$$ □

Theorem 3.2. *Let G be an n-vertex graph and*

$$\mathrm{V}(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \ne j,$$

where V_i are δ_k-small sets. Then for all integer $k \le r$ the following inequalities are satisfied

(i) $\mathrm{d}_k(G) \le \dfrac{n(r-1)}{r}$;

(ii) $r \ge \dfrac{n}{n - \mathrm{d}_k(G)}$.

Proof. Let $n_i = |V_i|$, $i = 1, 2, \ldots, r$. Then

$$\sum_{v \in \mathrm{V}(G)} d^k(v) = \sum_{i=1}^{r} \sum_{v \in V_i} d^k(v) \le \sum_{i=1}^{r} n_i (n - n_i)^k.$$

Let $\beta_i = 1 - \dfrac{n_i}{n}$, $i = 1, 2, \ldots, r$. Then

$$\sum_{v \in \mathrm{V}(G)} d^k(v) \le n^{k+1} \sum_{i=1}^{r} \beta_i (1 - \beta_i)^k, \quad k \ge r.$$

The inequality (i) follows from the last inequality and Lemma 3.1. Solving the inequality (i) for r, we derive the inequality (ii). □

4. Some corollaries from Theorem 3.2.

Corollary 4.1. *Let G be an n-vertex graph and let k and s be positive integers such that $k \le \varphi^{(s)}(G)$. Then*

(i) $\mathrm{d}_k(G) \le \dfrac{\big(\varphi^{(s)}(G)-1\big)n}{\varphi^{(s)}(G)} \le \dfrac{\big(\varphi(G)-1\big)n}{\varphi(G)} \le \dfrac{(\omega(G)-1)n}{\omega(G)} \le \dfrac{(\chi(G)-1)n}{\chi(G)};$

(ii) $\varphi^{(s)}(G) \ge \dfrac{n}{n-\mathrm{d}_k(G)}.$

Proof. Let $\varphi^{(s)}(G)=r$ and $\mathrm{V}(G)=V_1\cup V_2\cup\cdots\cup V_r$, $V_i\cap V_j=\emptyset$, where V_i are δ_k-small sets. Then the left inequality in (i) follows from Theorem 3.2 (i). The other inequalities in (i) follow from the inequalities $\varphi^{(s)}(G)\le\varphi(G)\le\omega(G)\le\chi(G)$. The inequality (ii) follows from Theorem 3.2 (ii). □

Remark 2. In the case $k=s=1$, Corollary 4.1 is proven in [1] (cf. Theorem 6.3 (i) and Theorem 6.2 (ii)).

Corollary 4.2. *Let G be an n-vertex graph. Then for all integer $s\ge 2$,*

$$\varphi^{(s)}(G) \ge \frac{n}{n-\mathrm{d}_2(G)}.$$

Proof. If $\varphi^{(2)}(G)=1$ then $\mathrm{E}(G)=\emptyset$, i.e. $G=\overline{K}_n$ and the inequality is obvious. If $\varphi^{(2)}(G)\ge 2$ then $\varphi^{(s)}(G)\ge 2$ because $s\ge 2$. Therefore Corollary 4.2 follows from Corollary 4.1 (ii). □

Corollary 4.3 ([2]). *For every n-vertex graph*

$$\varphi(G) \ge \frac{n}{n-\mathrm{d}_2(G)}.$$

Proof. This inequality follows from Corollary 4.2 because $\varphi^{(s)}(G)\le\varphi(G)$. □

Corollary 4.4 ([1]). *Let G be an n-vertex graph. Then for every positive integer $k\le\varphi(G)$*

$$\varphi(G) \ge \frac{n}{n-\mathrm{d}_k(G)}.$$

Proof. According to Theorem 2.1 there exists a natural number s such that $\varphi(G)=\varphi^{(s)}(G)$. Since $k\le\varphi^{(s)}(G)$ from Corollary 4.1 (ii) we derive

$$\varphi(G)=\varphi^{(s)}(G) \ge \frac{n}{n-\mathrm{d}_k(G)}. \qquad \square$$

Corollary 4.5. *Let G be an n-vertex graph. Then for every integer $s\ge 3$*

$$\varphi^{(s)}(G) \ge \frac{n}{n-\mathrm{d}_3(G)}.$$

Proof. Since $s\ge 3$, $\varphi^{(s)}(G)\ge\varphi^{(3)}(G)$. Therefore it is sufficient to prove the inequality

$$\varphi^{(3)}(G) \ge \frac{n}{n-\mathrm{d}_3(G)}. \tag{4.1}$$

If $\varphi^{(3)}(G)\ge 3$ then (4.1) follows from Corollary 4.1 (ii). If $\varphi^{(3)}(G)=1$ then the inequality (4.1) is obvious because $\mathrm{d}_3(G)=0$. Let $\varphi^{(3)}(G)=2$ and $\mathrm{V}(G)=V_1\cup V_2$, where V_i, $i=1,2$ are δ_3-small sets. Let $n_i=|V_i|$, $i=1,2$. Then

$$\sum_{v\in\mathrm{V}(G)} d^3(v) = \sum_{v\in V_1} d^3(v) + \sum_{v\in V_2} d^3(v) \le \tag{4.2}$$
$$n_1(n-n_1)^3+n_2(n-n_2)^3 = n_1n_2(n^2-2n_1n_2) \le \frac{n^4}{8}.$$

Therefore $\mathrm{d}_3(G) \le \frac{n}{2}$ and we obtain

$$\frac{n}{n-\mathrm{d}_3(G)} \le 2 = \varphi^{(3)}(G). \qquad \square$$

Since $\varphi(G) \ge \varphi^{(3)}(G)$ from Corollary 4.5 we derive

Corollary 4.6 ([1])**.** *For every n-vertex graph G*

$$\varphi(G) \ge \frac{n}{n-\mathrm{d}_3(G)}.$$

Corollary 4.7. *Let G be an n-vertex graph and $\varphi^{(4)}(G) \ne 2$. Then for every integer $s \ge 4$,*

$$\varphi^{(s)}(G) \ge \frac{n}{n-\mathrm{d}_4(G)}.$$

Proof. Since $\varphi^{(s)}(G) \ge \varphi^{(4)}(G)$ for $s \ge 4$, it sufficient to prove the inequality

$$\varphi^{(4)}(G) \ge \frac{n}{n-\mathrm{d}_4(G)}. \tag{4.3}$$

If $\varphi^{(4)}(G) \ge 4$ the inequality (4.3) follows from Corollary 4.1 (ii). If $\varphi^{(4)}(G) = 1$ the inequality (4.3) is obvious because $\mathrm{d}_4(G) = 0$. It remains to consider the case $\varphi^{(4)}(G) = 3$. Let $\mathrm{V}(G) = V_1 \cup V_2 \cup V_3$, where V_i, are δ_4-small sets and let $n_i = |V_i|$, $i = 1$, 2, 3. Then

$$\sum_{v\in \mathrm{V}(G)} d^4(v) = \sum_{v\in V_1} d^4(v) + \sum_{v\in V_2} d^4(v) + \sum_{v\in V_3} d^4(v) \le n_1(n-n_1)^4 + n_2(n-n_2)^4 + n_3(n-n_3)^4. \tag{4.4}$$

Denoting $\beta_i = 1 - \frac{n_i}{n}$, $i = 1, 2, 3$ we receive

$$\sum_{v\in \mathrm{V}(G)} d^4(v) \le n^4 \left(\sum_{i=1}^{3} (1-\beta_i)\beta_i^4 \right).$$

Since $\sum_{i=1}^{3} (1-\beta_i)\beta_i^4 \le \frac{2}{3}$ (see the proof of Theorem 5.4 (iii) in [1]) we take

$$\mathrm{d}_4(G) \le \frac{2}{3} = \frac{\varphi^{(4)}(G) - 1}{\varphi^{(4)}(G)}.$$

Solving the last equation for $\varphi^{(4)}(G)$ we obtain (4.3). $\square$

Corollary 4.8. *Let G be an n-vertex graph and $\varphi^{(4)}(G) \ne 2$. Then*

$$\varphi(G) \ge \frac{n}{n-\mathrm{d}_4(G)}. \tag{4.5}$$

Remark 3. In [1] it is proven that the inequlity (4.3) is held if $\varphi(G) \ne 2$.

5. Maximal $\boldsymbol{\delta_k}$-sets. We denote the maximal number of vertices in a δ_k-set of G by $\alpha^{(k)}(G)$. $S(G)$ is the maximal number of vertices of small sets of G. It is easilly seen that Proposition 1.3 yields.

Proposition 5.1. *For every graph G*

$$\alpha^{(1)}(G) \ge \alpha^{(2)}(G) \ge \cdots \ge \alpha^{(k)}(G) \ge \cdots \ge S(G) \ge \alpha(G).$$

Remark 4. Note that $\alpha^{(1)}(G)$ is denoted in [1] by $S^{\alpha}(G)$.

From Theorem 2.1 we have

Theorem 5.2. *For every graph G there exists an unique number $k_0 = k_0(G)$ such that*

$$\alpha^{(1)}(G) \geq \alpha^{(2)}(G) \geq \cdots \geq \alpha^{(k_0)}(G) = \alpha^{(k_0+1)}(G) \cdots = S(G).$$

Proposition 5.3. *Let $\mathrm{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. Then*

$$\begin{aligned}\alpha^{(k)}(G) &= \max\{s \mid \mathrm{d}_k(\{v_1, v_2, \ldots v_s\}) \leq n - s\} = \\ &= \max\{s \mid \{v_1, v_2, \ldots v_s\} \text{ is } \delta_k\text{-small set in } G\}.\end{aligned}$$

Proof. Let $s_0 = \max\{s \mid \{v_1, v_2, \ldots v_s\}$ be δ_k-small set in $G\}$. Then $s_0 \leq \alpha^{(k)}(G)$. Let $\alpha^{(k)}(G) = r$ and let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ be a δ_k-small set. Since $\mathrm{d}_k(\{v_1, v_2, \ldots, v_r\}) \leq \mathrm{d}_k(\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\})$ it follows that $\{v_1, v_2, \ldots, v_r\}$ is δ_k-small set too. Therefore $\alpha^{(k)}(G) = r \leq s_0$. □

Proposition 5.4. *For every positive number k the following inequalities hold*

$$n - \Delta(G) \leq \alpha^{(k)}(G) \leq n - \delta(G).$$

Proof. The left inequality follows from the inequality $S(G) \geq n - \Delta(G)$ from [1] and Proposition 5.1. Let $r = \alpha^{(k)}(G)$. According to Proposition 5.3, $\{v_1, v_2, \ldots, v_r\}$ is a δ_k-small set. So

$$\delta(G) = d(v_1) \leq \mathrm{d}_k(\{v_1, v_2, \ldots, v_r\}) \leq n - r = n - \alpha^{(k)}(G),$$

hence $\alpha^{(k)}(G) \leq n - \delta(G)$. □

Remark 5. The inequality $\alpha(G) \geq n - \Delta(G)$ is not always true. For example, $\alpha(C_5) < 5 - \Delta(C_5) = 3$.

Theorem 5.5. *Let $A \subseteq \mathrm{V}(G)$ be a δ_1-small set of G and $s = \mathrm{d}_1(\mathrm{V}(G) \setminus A)$. Then*

$$|A| \leq \left\lfloor \frac{n-s}{2} + \sqrt{\frac{(n-s)^2}{4} + ns - 2e(G)} \right\rfloor \tag{5.1}$$

Proof.

$$2e(G) = \sum_{v \in \mathrm{V}(G)} d(v) = \sum_{v \in A} d(v) + \sum_{v \in \mathrm{V}(G) \setminus A} d(v) \leq |A|\,(n - |A|) + s(n - |A|).$$

Solving for $|A|$ we obtain the inequality (5.1). □

Corollary 5.6 ([1])**.** *For every number k*

$$\begin{aligned}\alpha^{(k)}(G) &\leq \left\lfloor \frac{n - \Delta(G)}{2} + \sqrt{\frac{(n - \Delta(G))^2}{4} + n\Delta(G) - 2e(G)} \right\rfloor \leq \\ &\leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor.\end{aligned} \tag{5.2}$$

Proof. According to Proposition 5.1, it is sufficient to prove (5.2) only in the case $k = 1$. Let A be a maximal δ_1-small set, i.e. $|A| = \alpha^{(1)}(G)$, and $s = \mathrm{d}_1\left(\mathrm{V}(G) \setminus A\right)$. According to Theorem 5.5 the inequality (5.1) holds. Since the right side of (5.1) is an increasing function for s and $s \leq \Delta(G) \leq n - 1$, the inequalities (5.2) follow from (5.1). □

6. α-small sets.

Definition 3 ([1])**.** Let G be an n-vertex graph and let $W \subseteq \mathrm{V}(G)$. We say that W

is an α-small set if

$$\sum_{v \in W} \frac{1}{n - d(v)} \leq 1.$$

We denote the smallest natural number r for which $\mathrm{V}(G)$ decomposes into r α-small sets by $\varphi^{\alpha}(G)$.

The idea for α-small sets is coming from the following Caro-Wey inequality ([3] and [7])

$$\omega(G) \geq \sum_{v \in \mathrm{V}(G)} \frac{1}{n - d(v)}.$$

We have the proposition

Proposition 6.1 ([1]).

$$\varphi^{(1)}(G) \leq \varphi^{\alpha}(G) \leq \varphi(G).$$

The following problem is inspired by Proposition 6.1 and Theorem 2.1.

Problem. Is it true that for every graph G there exists natural number $k_0 = k_0(G)$ such that $\varphi^{\alpha}(G) = \varphi^{(k_0)}(G)$?

REFERENCES

[1] A. Bojilov, Y. Caro, A. Hansberg, N. Nenov. Partitions of graphs into small and large sets. *Discrete Applied Mathematics* (to appear), arXiv:1205.1727.

[2] A. Bojilov, N. Nenov. An inequality for generalized chromatic graphs. *Math. and Education in Math.*, **41** (2012), 143–147.

[3] Y. Caro. New results on the independence number. Tech. report, Tel-Aviv University, 1979.

[4] G. H. Hardy, J. F. Litelewood, G. Polya. Inequalities, 1934.

[5] N. Khadzhiivanov. Extremal theory of graphs. Sofia, Sofia University, 1990, (in Bulgarian).

[6] N. Nenov. Improvement of graph theory Wei's inequlity. *Math. and Education in Math.*, **35** (2006), 191–194.

[7] V. K. Wei. A lower bound on the stability number of a simple graph. Technical Memorandum 81–11217–9, Bell Laboratories, Murray Hill, NJ, 1981.

[8] D. B. West. Introduction to graph theory, second ed.. Prentice Hall, Inc., Upper Saddle River, NJ, 2001, xx+588 pp.

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δ_K-МАЛКИ МНОЖЕСТВА В ГРАФИ

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Нека G е прост n-върхов граф и $W \subseteq \mathrm{V}(G)$. Казваме, че W е δ_k-малко множество, ако

$$\sqrt[k]{\frac{\sum_{v\in W} \mathrm{d}^k(v)}{|W|}} \leq n - |W|\,.$$

$\varphi^{(k)}(G)$ означава най-малкото естествено число r, за което $\mathrm{V}(G)$ се разлага на r δ_k-малки множества, а $\alpha^{(k)}(G)$ означава максимума на броя на върховете на δ_k-малките множества на G. В тази работа ние получаваме оценки за $\alpha^{(k)}(G)$ и $\varphi^{(k)}(G)$. Тъй като $\varphi^{(k)}(G) \leq \omega(G) \leq \chi(G)$ и $\alpha(G) \leq \alpha^{(k)}(G)$, получаваме също оценки за кликовото число $\omega(G)$, хроматичното число $\chi(G)$ и числото на независимост $\alpha(G)$.