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EXTENSIONS OF BRAIDED GROUPS*

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Set-theoretic solutions of the Yang–Baxter equation form a meeting-ground of mathematical physics, algebra and combinatorics. Such a solution (X, r) consists of a set X and a bijective map $r : X \times X \to X \times X$ which satisfies the braid relations. Braided groups and symmetric groups (involutive braided groups) are group analogues of braided sets and symmetric sets. They are important for the theory of set-theoretic solutions of the Yang-Baxter equation. We introduce *a regular extension* of braided (respectively, symmetric) groups S, T as a braided (resp: symmetric) group (U, r) such that $U = S \bowtie T$ is the double cross product of S and T, where (S, T) is a strong matched pair and the actions of (U, U) extend the actions of S and T. We study how the properties of the extension $U = S \bowtie T$ depend on the properties of S and T.

1. Introduction.

1.1. Matched pairs of groups, braided groups, and symmetric groups. The theory of matched pairs of groups was introduced and developed by Majid and Takeuchi, [6, 9].

A matched pair of groups is a triple (S, T, σ), where S and T are groups and $\sigma : T\times S \longrightarrow S\times T, \quad \sigma(a, u) = ({}^a u, a^u)$ is a bijective map satisfying the following conditions $\forall\, a, b \in T, u, v \in S$:

$$\mathbf{ML0}: \quad {}^a 1 = 1,\ {}^1 u = u, \quad \mathbf{ML1}: \quad {}^{ab}u = {}^a({}^b u), \quad \mathbf{ML2}: \quad {}^a(u.v) = ({}^a u)({}^{a^u} v),$$
$$\mathbf{MR0}: \quad 1^u = 1,\ a^1 = a, \quad \mathbf{MR1}: \quad a^{uv} = (a^u)^v, \quad \mathbf{MR2}: \quad (a.b)^u = (a^{{}^b u})(b^u).$$

In other words, the group T acts upon S from the left by ${}^{(\,)}\bullet$, (**ML0**, **ML1**), S acts on T from the right by $\bullet^{(\,)}$ (**MR0**, **MR1**) and these two actions obey the conditions **ML2** and **MR2**.

A braided group is a pair (G, σ), where G is a group and $\sigma : G \times G \longrightarrow G \times G$ is a map such that the triple (G, G, σ) forms a matched pair of groups, and the left and the right actions induced by σ satisfy *the compatibility condition*, see [8]:

$$\mathbf{M3}: \quad uv = ({}^u v)(u^v), \text{ is an equality in } G \quad \forall u, v \in G.$$

If the map σ is involutive ($\sigma^2 = id_{G\times G}$) then (G, σ) is called a *symmetric group*, see [6]. Braided groups and symmetric groups are group analogues of braided sets and symmetric sets introduced in [2].

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A matched pair (S,T) of groups implies the existence of a group $S \bowtie T$ (called the *double cross product*) built on $S \times T$ with product and unit

$$(u,a)(v,b) = (u.{}^{a}v, a^{v}.b), \quad 1 = (1,1), \quad \forall u,v \in S,\ a,b \in T$$

and containing S,T as subgroups. Conversely, suppose that there exists a group R factorising into subgroups S,T in the sense that (i) $S,T \subseteq R$ are subgroups and (ii) the restriction of the product of R to a map $\mu : S \times T \to R$ is bijective. Then (S,T) form a matched pair and $R \cong S \bowtie T$ by this identification μ.

A *strong group factorisation* is a factorisation in subgroups S,T as above such that R also factorises into T,S. We say that a matched pair (S,T,σ) is *strong* if it corresponds to a strong factorisation.

Definition 1.1. *A regular extension of braided (respectively, symmetric groups) S,T is a braided (respectively, a symmetric) group (U,r) such that $U = S \bowtie T$ where (S,T) is a strong matched pair and the actions of (U,U) extend the actions of (S,S), (T,T), (S,T), (T,S). We denote the last of these by $\triangleleft, \triangleright$.*

If the actions in the initial matched pairs extend, then

$${}^{v}(u.a) = {}^{v}u((v^{u}) \triangleleft a), \quad {}^{b}(u.a) = {}^{b}u.{}^{b^{u}}a$$

are the only possible definitions for the actions of S,T. Hence the extended actions necessarily take the form

$${}^{(v.b)}(u.a) = {}^{v}({}^{b}u).((v^{{}^{b}u}) \triangleleft ({}^{b^{u}}a)), \quad (v.b)^{(u.a)} = ((v^{{}^{b}u}) \triangleright ({}^{b^{u}}a)).(b^{u})^{a}.$$

An argument similar to the proof of our analogous results for monoids, see Theorem 4.31, and Corollary 4.32 [5], verifies the following theorem.

Theorem 1.2. *Let $U = S \bowtie T$, where (S,T) is a strong matched pair of braided groups (S,r_S) and (T,r_T). The following are equivalent:*

(1) *U is a regular extension of braided groups. In particular, (U,σ) is a braided group, where the braiding operator σ extends r_S and r_T.*

(2) *$(U,T),(S,U)$ are matched pairs extending the given actions.*

(3) *The following equalities hold for all $u,v \in S$ and $a,b \in T$:*

$$\begin{array}{lll} & \mathbf{ml1a}: \quad {}^{{}^{a}u}({}^{a^{u}}v) = {}^{a}({}^{u}v), & \mathbf{lr3a}: \quad ({}^{a}u)^{({}^{a^{u}}v)} = {}^{(a^{{}^{u}v})}(u^{v}) \\ (1.1) & & \\ & \mathbf{mr1a}: \quad (a^{b})^{u} = (a^{{}^{b}u})^{b^{u}}, & \mathbf{lr3b}: \quad ({}^{a}b)^{({}^{a^{b}}u)} = {}^{(a^{{}^{b}u})}(b^{u}). \end{array}$$

(4) *Moreover, if (S,r_S) and (T,r_T) are symmetric groups, i.e. $(r_S)^2 = 1$, $(r_T)^2 = 1$, then $U = S \bowtie T$ is also a symmetric group.*

Problem 1.3. *Let $U = S \bowtie T$ be a regular extension of symmetric groups.* Given *that (S,r_S) and (T,r_T) satisfy some special condition, say: (i) (S,r_S) and (T,r_T) have finite multipermutation levels; (ii) S and T satisfy **lri**, (iii) S and T satisfy **Raut**, (iv) $(S,+,\cdot)$ and $(T,+,\cdot)$ are two-sided braces.* Decide *when U inherits the same condition. In particular, study the case of particular constructions like semidirect products, or wreath products of symmetric groups.*

2. Semidirect products of symmetric groups. Let (S,r_S) and (T,r_T) be disjoint symmetric groups. Suppose the group T acts on the right on S as automorphisms, that is there is a group homomorphism $\varphi : T \longrightarrow \mathrm{Aut}(S, r_S)$. Denote the right action $u.\varphi(a) = u^a$, then one has

$$u^{(ab)} = (u^a)^b \quad (uv)^a = u^a.v^a, \forall u, v \in S, a, b \in T.$$

Consider the outer semidirect product of S and T, (with respect to φ), denoted $U = S \rtimes_\varphi T = S \rtimes T$. As a set, $S \rtimes T$ is the cartesian product $S \times T$. Multiplication of elements in $S \rtimes T$ is determined by the right action as

$$(u, a).(v, b) := (u({}^a v), ab), \quad \forall u, v \in S, a, b \in T.$$

This defines a group in which the identity element is $(e_S, e_T) = (1, 1)$ and the inverse of the element (u, a) is $({}^{a^{-1}}(u^{-1}), a^{-1})$. The set of pairs $(u, 1)$ form a normal subgroup of U isomorphic to S, while pairs $(1, a)$ form a subgroup isomorphic to T. The full group U is an inner semidirect product of those two subgroups, so we shall use notation $ua := (u, a)$. There is a canonical structure of symmetric group on U, moreover, U is a regular extension of S, T. Suppose there is a (nondegenerate) left action of T on S, such that U is a symmetric group. Let $u \in S, a \in T$. The compatibility condition and the multiplication law in U imply

$$ua = {}^u a.u^a = a.u^a = ({}^a(u^a))a,$$

hence,

$${}^a(u^a) = u = {}^a({}^{a^{-1}}u), \quad \forall u \in S, a \in T,$$

which by the nondegeneracy of the action gives:

$$u^a = {}^{a^{-1}}u, \quad \forall u \in S, a \in T.$$

This equality determines canonically the left action of T on S, as

$${}^a u := u^{a^{-1}}, \forall u \in S, a \in T,$$

It is easy to check that T acts on the left upon S as automorphisms, and the identities (1.1) are satisfied. Hence the bijective map $r : U \times U \longrightarrow U \times U$ defined as $r(ua, vb) = ({}^{ua}(vb), (ua)^{vb})$ is an involutive braiding operator on U, and (U, r) is a symmetric group.

We say that $(U, r) = (S \rtimes T, r)$ is *a semidirect product of the symmetric groups S, T*

It will be interesting to describe the socle Γ_U of U. By definition, Γ_U is kernel of the left actions

$$\Gamma_U = \Gamma_l = \{x \in U \mid {}^x y = y, \ \forall y \in U\} = \{x \in U \mid y^x = y, \ \forall u \in G\} = \Gamma_r,$$

Denote by Γ_S, and Γ_T, respectively the corresponding kernels in S, T, respectively. Recall that Γ_U is an r-invariant normal subgroup of U, the quotient group $\widetilde{G} = U/\Gamma$ is a symmetric group, and the map

$$\varphi : (\widetilde{U}, r_{\widetilde{U}}) \longrightarrow ([U], r_{[U]}) = \mathrm{Ret}(U, r), \quad \widetilde{a} \mapsto [a],$$

is an isomorphism of symmetric groups, as shows the following.

Lemma 2.1. *Suppose $(U, r) = S \rtimes_\varphi T = (S \rtimes T, r)$ is the semidirect product of symmetric groups defined as above, let Γ_U, Γ_S, Γ_T be the corresponding socles.*

(1) *The socle Γ_S is an r-invariant normal subgroup of U. Moreover, $\Gamma_S \subseteq \Gamma_U$, and $(S, r_S) \ncong \mathrm{Ret}(S, r_S)$ implies $(U, r) \ncong \mathrm{Ret}(U, r)$.*

(2) *The socle* Γ_U *is a union*

$$\Gamma_U = \Gamma_S \bigcup \{x \in U | x = ua, u \in S, \ (\textit{possibly}\ u = 1), \ a \in \Gamma_U, \\ \mathcal{L}_{a|S} = \mathcal{L}_{u^{-1}} \in \mathrm{Aut}(S, r_S)\}.$$

(3) *The original action of* T *on* S *induces an action of* T *on the quotient group* $S/\Gamma_S \simeq ([S], r_{[S]})$*, and a semidirect product* $(S/\Gamma_S) \rtimes T$ *. The following is a sequence of surjective homomorphisms of symmetric groups:*

$$U = S \rtimes T \longrightarrow U/\Gamma_S \simeq (S/\Gamma_S) \rtimes T \longrightarrow U/\Gamma_U \simeq \mathrm{Ret}(U, r) \quad ua \mapsto ([u]_S)a \mapsto [ua]_U.$$

Proof. First we show that Γ_S is a normal subgroup of U. Let $x \in U, u \in \Gamma_S$, then for every $v \in S$ one has

$$^{(xux^{-1})}v = {}^{x}({}^{(ux^{-1})}v) = {}^{x}({}^{x^{-1}}v) = v,$$

therefore $xux^{-1}) \in \Gamma_S$. Next we verify that Γ_S is r-invariant, or equivalently, it is invariant with respect to the left and to the right actions of U. We know that Γ_S is invariant with respect to the action of S upon itself, so it will be enough to show that

$$^{a}u \in \Gamma_S, u^a \in \Gamma_S \ \forall a \in T, u \in \Gamma_S.$$

Let $a \in T, u \in \Gamma_S$, we must show $^{^a u}v = v, \forall v \in S$. Let $v \in S$, and set $w = v^a$. Then $v = {}^a w$,

$$^{^a u}v = {}^{^a u}({}^a w) = {}^{(^a u)a}w = {}^{au}w = {}^a w = v.$$

So Γ_S is invariant with respect to the left action of U, and the equality $u^a = {}^{a^{-1}}u \in \Gamma_S$ implies Γ_S is invariant with respect to the right action of U. This proves part (1) It follows that the canonical maps

$$U = S \rtimes T \longrightarrow U/\Gamma_S \longrightarrow U/\Gamma_U$$

are epimorphisms of symmetric groups. Moreover, there is a natural isomorphism of symmetric groups $U/\Gamma_S \simeq (S/\Gamma_S) \rtimes T$. $\square$

Theorem 2.2. *Let* (S, r_S) *and* (T, r_T) *be disjoint symmetric groups, such that* T *acts on* S *as automorphisms. Suppose* $(U, r) = (S \rtimes T, r)$ *is the semidirect product of symmetric groups defined as above.*

(1) (U, r) *is a symmetric group which is a regular extension of* S, T*.*

(2) (U, r) *satisfies* ***lri*** iff (S, r_S) *and* (T, r_T) *satisfy* ***lri****.*

(3) (U, r) *satisfies condition* ***Raut*** iff (S, r_S) *and* (T, r_T) *satisfy* ***Raut****.*

(4) (U, r) *has finite multipermutation level if and only if* (S, r_S) *and* (T, r_T) *do so. In this case the following inequalities give exact bounds for* $\mathrm{mpl}(U, r)$ *:*

$$\max\{\mathrm{mpl}\, S, \mathrm{mpl}\, T\} \leq \mathrm{mpl}\, U \leq \mathrm{mpl}\, S + \mathrm{mpl}\, T.$$

(5) *The left brace (U,+, .) is a two-sided brace if and only if* S *and* T *are two-sided braces and*

$$(u + v)^{a+b}w + w = u^a w + v^b w \quad \forall u, v, w \in S, |; a, b \in T.$$

3. Wreath products of symmetric groups. Let A and H be (disjoint) groups and suppose H acts on a set Ω. We recall the definition of (restricted) wreath product $Awr_\Omega H$. Let K be the direct sum

$$K \equiv \bigoplus_{\omega \in \Omega} A_\omega$$

of copies of $A_\omega \simeq A$ indexed by the set Ω. The elements of K are sequences (a_ω) of elements in A indexed by Ω of which all but finitely many (a_ω) are the identity element of A. Then the action of H on Ω extends in a natural way to an action of H on the group K by

$${}^h(a_\omega) = (a_{{}^h\omega}).$$

Then the wreath product $Awr_\Omega H$ of A by H is the semidirect product $K \rtimes H$. The subgroup K of $Awr_\Omega H$ is called the base of the wreath product.

Assume now (A, r_A) and (H, r_H) are symmetric groups. In this case H acts on the left and on the right upon itself. We consider the wreath product $AwrH$, where $\Omega := H$.

Note that the left and the right action of A upon itself induce in a natural way a left and a right action of K upon itself which makes K a symmetric group. The left and the right actions are define componentwise as:

$${}^{(a_\alpha)}b_\beta = \begin{cases} ({}^a b)_\alpha & \text{if} \quad \beta = \alpha \\ b_\beta & \text{else.} \end{cases}$$

The right action of K upon itself is defined analogously. Then the braiding operator r_K is defined canonically as

$$r_K((a_\alpha)(b_\beta)) = ((({}^a b)_\alpha), ((a^b)_\alpha))$$

Theorem 3.1. *Let (A, r_A) and (H, r_H) be disjoint symmetric groups. Then the wreath product $G = AwrH$ is a symmetric group.*

Let (X_0, r_{X_0}) and (Y, r_Y) be disjoint square-free solutions. The wreath product of solutions, denoted $(Z, r) = (X_0, r_{X_0}) \wr (Y, r_Y)$ is defined in [7, Definition 8.6].

Theorem 3.2. *Let (X_0, r_0) and (Y, r_Y) be disjoint square-free solutions, and let $(Z, r) = (X_0, r_0) \wr (Y, r_Y)$ be their wreath product. As usual, $\mathcal{G}_{X_0} = \mathcal{G}(X_0, r_0)$, $\mathcal{G}_Y = \mathcal{G}(Y, r_Y)$ and $\mathcal{G}_Z = \mathcal{G}(Z, r)$ denote the corresponding permutation groups, we consider also the corresponding symmetric group structure on each of them. Then the following conditions hold.*

(1) *The wreath product $(Z, r) = (X_0, r_0) \wr (Y, r_Y)$ is a square-free solution.*

(2) *The permutation group $\mathcal{G}_Z$ is a wreath product of the groups $\mathcal{G}_Z = \mathcal{G}_{X_0} \wr \mathcal{G}_Y$.*

(3) *Moreover, the symmetric group $(\mathcal{G}_Z, r_{\mathcal{G}_Z})$ is a wreath product of symmetric groups:*

$$(\mathcal{G}_Z, r_{\mathcal{G}_Z}) = (\mathcal{G}_{X_0}, r_{\mathcal{G}_{X_0}}) \wr (\mathcal{G}_Y, r_{\mathcal{G}_Y})$$

Suppose (X_0, r_0) and (Y, r_Y) are multipermutation solutions of finite multipermutation levels, $\operatorname{mpl} X_0 = m$, and $\operatorname{mpl} Y = n$.

(4)

$$\operatorname{mpl} Z = \operatorname{mpl} X_0 + \operatorname{mpl} Y - 1 \quad \operatorname{mpl} \mathcal{G}_{X_0} = m - 1, \operatorname{mpl} \mathcal{G}_Y = n - 1.$$

(5)

$$\operatorname{mpl}\mathcal{G}_Z = \operatorname{mpl}\mathcal{G}_{X_0} + \operatorname{mpl}\mathcal{G}_Y = \operatorname{mpl} Z - 1$$

Proof. Parts (3.2) and (3.2) are proven in [7, Theorem 8.7]. Parts (3.2) is straightforward. By [4, Theorem 5.15] there follow the equalities $\operatorname{mpl}\mathcal{G}_{X_0} = \operatorname{mpl} X_0 - 1, \operatorname{mpl}\mathcal{G}_Y = n-1$ and $\operatorname{mpl}\mathcal{G}_Z = \operatorname{mpl} Z - 1$. Moreover, [7, Theorem 8.7] implies $\operatorname{mpl} Z = \operatorname{mpl} X_0 + \operatorname{mpl} Y - 1$. It follows that

$$\operatorname{mpl}\mathcal{G}_Z = \operatorname{mpl} Z - 1 = m + n - 2 = \operatorname{mpl}\mathcal{G}_{X_0} + \operatorname{mpl}\mathcal{G}_Y. \qquad \square$$

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РАЗШИРЕНИЯ НА СПЛЕТЕНИ ГРУПИ И СИМЕТРИЧНИ ГРУПИ

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Уравнението на Янг-Бакстер е едно от основните уравнения на математическата физика, и по-специално – в теорията на квантовите групи. Особено интересни са теоретико-множествените решения, при чието изследване освен теоретична физика се използват интензивно некомутативна алгебра и комбинаторика. Такова решение (X, r) се състои от множество X и биективно изображение $r : X \times X \to X \times X$, което удовлетворява известното *съотношение на плитките*. Сплетените групи и симетричните групи (т.е. сплетените инволютивни групи) са теоретико-групови аналози на т.н. *сплетени множества* и *симетрични множества*. Модерните тенденции налагат интензивно изучаване на теорията на сплетените и симетричните групи. Всяка такава група представлява и решения на уравнението на Янг-Бакстър. Конструирането на нови решения (нови симетрични групи) е от особена важност. Ние въвеждаме понятието „регулярни разширения на сплетени групи S и T“. И изследваме кои алгебрични свойства на S и T индуцират аналогични свойства на техни разширения $U(S, T)$.