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POHOŽAEV IDENTITIES AND APPLICATIONS TO MIXED TYPE EQUATIONS OF MATHEMATICAL PHYSICS*

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Dedicated to the 80-th anniversary of professor A. M. Nakhushev

For nonlinear equations of Gellerstedt type, uniqueness of regular solution to the degenerate hyperbolic Cauchy-Goursat problem on triangular domain with sub-characterstic will be established. For homogeneous *supercritical* nonlinearities, the uniqueness of the trivial solution in the class of regular solutions will be approved by combining suitable Pohožaev-type identities with appropriate assumptions for the domain and application of Hardy-Sobolev inequality. Introduction to some open questions regarding generalized solvability of the problem will be given.

1. Introduction. ***The Sobolev embedding theorem.*** Let Ω be a bounded smooth domain in $\mathbb{R}^n, n \in N,\ n \geq 3$. Then one has the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ with $q \leq \dfrac{2n}{n-2}$. The critical Sobolev exponent is denoted by $2^*(n) := \dfrac{2n}{n-2}$ and the embedding is compact for $q \in [1, 2^*(n))$, but fails to be compact at the so called *critical case* when $q = 2^*(n)$.

It is well known, starting from the seminal paper of Pohožaev [11], that the homogeneous Dirichlet problem for semi-linear elliptic equations such as $\Delta u + u|u|^{p-2} = 0$ will permit only the trivial solution $u \equiv 0$, if the domain is star-shaped, the solution is sufficiently regular, and $p > 2^*(n) = 2n/(n-2)$.

On the other hand, at subcritical growth $2 < p < 2^*(n)$ in the nonlinearity such as $F'(u) = u|u|^{p-2}$, one generically does have existence of nontrivial solutions. Let's mention explicitly the critical case $p = 2^*(n)$, when the situation is quite more interesting. Beginning with the celebrated paper of Brezis and Nirenberg [1] a lot of works with more precise results are given in this case.

Pohožaev-type identities have been used in a large number of papers, mostly related to elliptic problems. We will just mention here the works of Pucci, Serrin [13], D'Ambrosio, Mitidieri [3] and Monticelli, Payne and Punzo [9] which treat elliptic or degenerate elliptic cases.

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Recent studies have extended Pohožaev-type identities to nonlocal problems (see for example Ros-Oton, Serra [14]). General formulation of the nonlocal problem is

$$(-\Delta)^s u = f(u) \text{ in } D,$$
$$u = 0 \text{ in } \mathbb{R}^n \setminus D,$$

where $D \subset \mathbb{R}^n$ and $(-\Delta)^s$ is the fractional Laplacian defined for $0 < s < 1$ as

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.$$

A surprising fact in this case is that from a nonlocal problem, using a non-standard multiplier operator, one obtains a completely local boundary term in the Pohožaev identity. Although the function u has to be defined in the whole $\mathbb{R}^n$ in order to compute its fractional Laplacian at a given point, knowing u only in a neighborhood of the boundary one can already compute the boundary integral (see [14]).

Let us mention also some initial problems for Tricomi-type equations in the whole space (for example He, Witt, Yin [5]).

Beginning with paper [6], D. Lupo and K. Payne in a series of papers have also studied some classical boundary value problems for critical exponent phenomena for a class of mixed type elliptic-hyperbolic equations in two dimensions. **In the supercritical case**, a **nonexistence principle** was established in [6] for Tricomi type equations in two dimensions with suitable boundary conditions compatible with linear solvability theory. Later, in [8] Goursat boundary value problems for Tricomi equation had been studied establishing uniqueness results for generalized solvability in the supercritical and in the critical case, but also some existence results. Uniqueness results in the frame of generalized solvability for supercritical and critical growth were established in [12] further developing the results obtained in [8] to the Cauchy-Goursat problem for nonlinear Gellerstedt equations. In particular, the problems in [8] and [12] were considered in the domain $D = ABC$, a characteristic triangle:

$$D = \left\{ (x,y) \in \mathbb{R}^2, y \leq 0 : -2x_0 + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} \leq x \leq -\frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \right\}. \tag{1}$$

This is a connected region in the plane whose boundary consists of the segment AB, $A = (-2x_0, 0), B = (0,0)$ on the $x - axis$ and two characteristic arcs AC and BC.

The point of this work is to continue our study of two-dimensional problems for Gellerstedt equation started in [8] and continued in [12]. Specifically, we study the semilinear Cauchy-Goursat problem

$$Lu + F'(u) = 0 \text{ in } \Omega, \tag{2}$$

$$u = 0 \text{ on } AC, \ u_y = 0 \text{ on } AB, \tag{3}$$

where $L = -(-y)^m D_x^2 + D_y^2$, $m > 0$ is a Gellerstedt operator on $\mathbb{R}^2$ and $F'(u) = u|u|^{p-2}$ is a power type nonlinearity. The region $\Omega = ABC$ is a triangular domain similar to D given by (1), but here the characteristic arc AC is changed, namely it is replaced by a sub-characteristic arc Γ, starting from point A reaching a point which we still call $C(x_C, y_C)$, on the characteristic of positive slope through B. On the sub-characteristic Γ the following inequality holds

$$-(-y)^m \nu_x^2 + \nu_y^2 \geq 0 \tag{4}$$

where $\nu = (\nu_x, \nu_y)$ is the external unit normal vector to the boundary. If the inequality (4) holds in the strict sense on some piece Γ' of Γ, we call Γ' strictly sub-characteristic which means that Γ' is a piece of spacelike surface for the operator L. The boundary conditions are placed on a proper subset of the boundary, namely on the boundary AC we have $u = 0$ and on the segment AB we have Neumann boundary condition $u_y = 0$. No data is given on the characteristic BC.

We say that the domain Ω is *M-star-shaped* if for every point $(x_0, y_0) \in \overline{\Omega}$ the flow generated along a Lipschitz continuous vector field $M = (m+2)xD_x + 2yD_y$ on $\mathbb{R}^2$ lies in $\overline{\Omega}$ for every moment $t \in [0, +\infty)$. If Ω is *M-star-shaped* we say that $\partial\Omega$ is *M-star-like* in the sense that on $\partial\Omega$ one has

$$((m+2)x, 2y) \cdot \nu \geq 0, \tag{5}$$

where ν is the external unit normal on $\partial\Omega$. If the inequality holds on a part S of the boundary, we say that S is *M-star-like*, or S is strictly *M-star-like* if the inequality is strict.

We define the weighted Sobolev space $H_0^1(\Omega; m)$ with norm

$$\|u\|^2_{H_0^1(\Omega;m)} := \int_\Omega \left(|y|^m u_x^2 + u_y^2\right) dxdy.$$

This is a natural norm for which to begin the search for weak solutions [7]. From the embedding of $H_0^1(\Omega; m)$ into $L^q(\Omega)$ the critical Sobolev exponent arises

$$2^*(1, m) = \frac{2(m+4)}{m}.$$

We will consider the problem (2), (3) in the case of the supercritical growth

$$p \geq 2^*(1, m) = \frac{2(m+4)}{m}, \tag{6}$$

and we will prove that in this case the only solution $u \in C^2(\overline{\Omega})$ is the trivial solution $u = 0$. For Tricomi equation (i.e. equation (2) with $m = 1$) and boundary conditions $u = 0$ *on* $AC \cup AB$ the case $p > 2^*(1, m)$ was treated in [6] and the extension to $p \geq 2^*(1, m)$ was treated in [7] for mixed type domains. In the frame of generalized solvability the case $p \geq 2^*(1, 1) = 10$ was announced in [4] and treated in [8].

2. Nonlinear theory: Pohožaev identities. At first we need to formulate the well known Hardy-Sobolev inequality with remainder term. It will be used when discussing the sign of the boundary integral under BC where no boundary conditions are imposed and is needed to ensure that the contribution of this integral has the right sign.

Lemma 1. *Let* $u_1 \in C^1[0, a]$ *and satisfy* $u_1(a) = 0$. *Let* $p > 1$, *then the following inequality holds*

$$\int_0^a t^p (u_1')^2 dt \geq \frac{(p-1)^2}{4} \int_0^a t^{p-2} u_1^2 dt + \frac{4}{a^2} \int_0^a t^p u_1^2 dt. \tag{7}$$

The inequality is a weighted Sobolev inequality which applies to classes of absolutely continuous functions. General approach for proof of the inequality could be found in Chen, Shen [2]. Slight modification of the proof with regard to the application in our particular case could be found in [8] and [12].

We can now formulate the theorem for non-existence of nontrivial regular solution for the problem (2)–(3).

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a triangular domain with boundary $AB \cup BC \cup AC$ with AC sub-characteristic in the sense (4). Assume that the noncharacteristic part of AC is star-like with respect to the vector field*

$$M = (m+2)xD_x + 2yD_y \ . \tag{8}$$

Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (2)–(3) with $F'(u) = u|u|^{p-2}$. Then $u \equiv 0$ in the supercritical case $p > 2^*(1,m)$, where $2^*(1,m) = 2(m+4)/m$ the critical Sobolev exponent. If in addition the noncharacteristic part of AC is strictly star-like, then the result holds also in the critical case $p = 2^*(1,m)$.

Proof. We begin the proof by multiplying the equation with considered vector field Mu, integrating over Ω and taking into account that the primitive F satisfies $F(0) = 0$, we obtain

$$\begin{aligned} &\int_\Omega \left[-(-y)^m u_{xx} + u_{yy} + F'(u)\right]\left[(m+2)xu_x + 2yu_y\right] = \\ &= \int_\Omega \left[\frac{m}{2}uF'(u) - (m+4)F(u)\right] dxdy + \\ &+ \int_{AC} W_1 \cdot \nu ds + \int_{BC} (W_1 + W_2 + W_3) \cdot \nu ds + \int_{AB} (W_1 + W_3) \cdot \nu ds \end{aligned} \tag{9}$$

where

$$W_1 = ((m+2)xu_x + 2yu_y)\left[-(-y)^m u_x, u_y\right] - \frac{1}{2}\left(-(-y)^m u_x^2 + u_y^2\right)\left[(m+2)x, 2y\right],$$

$$W_2 = \frac{m}{2}u\left(-(-y)^m u_x, u_y\right), \qquad W_3 = \left[(m+2)x, 2y\right]F(u).$$

Remark. The problem considered in [12] is similar but the difference here will be the additional integral over AC which is arising due to the noncharacteristic part of that boundary. Similar boundaries were considered in [7] for mixed type domains.

On AB we have conditions $u_y = 0$, $y = 0$ and from the fact that the external unit normal vector on the same boundary has components $\vec{\nu} = (0,1)$ it is obvious that the integral over AB vanishes.

The characteristic BC is given by

$$BC: \quad x + g(y) = 0, \quad y \in [y_C, 0], \quad g(y) = \frac{2}{m+2}(-y)^{\frac{m+2}{2}}. \tag{10}$$

Referring to (10) we derive the components of the external unit normal vector associated to BC

$$\vec{\nu}_{BC} = \frac{1}{[1 + (-y)^m]^{\frac{1}{2}}}\left(1, -(-y)^{\frac{m}{2}}\right).$$

We could directly observe that the vectors $((m+2)x, 2y)$ and $\vec{\nu}_{BC}$ are orthogonal. Parameterizing BC from (10) by $\beta(t) = (-g(t), t)$ with $t \in [y_C, 0]$ and setting $w(t) = u(\beta(t))$, one finds that the boundary integral in (13) is

$$\begin{aligned} &\int_{BC} (W_1 + W_2 + W_3) \cdot \nu ds = \int_{BC} \left[Mu + \frac{m}{2}u\right]\left(-(-y)^m u_x, u_y\right) \cdot \nu \, ds = \\ &= \int_{y_C}^0 \left[4(-t)^{\frac{m+2}{2}} w'(t)^2 - \frac{m^2}{4}(-t)^{\frac{m-2}{2}} w(t)^2\right] dt \geq \frac{16}{y_C}\int_{y_C}^0 (-t)^{\frac{m+2}{2}} w(t)^2 dt \geq 0 \end{aligned} \tag{11}$$

where we use Hardy-Sobolev inequality with remainder term (7).

Finally, on AC the boundary condition $u = 0$ implies that $u_x = u_\nu \nu_x$ and $u_y = u_\nu \nu_y$ on AC where u_ν is the normal derivative. Also, recalling that the noncharacteristic part of AC is *M-star-like* and sub-characteristic, using (4) and (5) we obtain the inequality

$$\int_{AC} W_1 \cdot \nu ds = \int_{AC} \frac{1}{2} u_\nu^2 [-(-y)^m \nu_x^2 + \nu_y^2][(m+2)x\nu_x + 2y\nu_y] ds \geq 0. \tag{12}$$

Thus combining results (9), (11) and (12) we obtain the following Pohožaev type identity

$$\begin{aligned} \int_\Omega Mu(Lu + F'(u)) dxdy &= \int_\Omega \left[\frac{m}{2} uF'(u) - (m+4)F(u)\right] dxdy + \\ &+ \int_{AC} \frac{1}{2} u_\nu^2 [-(-y)^m \nu_x^2 + \nu_y^2][(m+2)x\nu_x + 2y\nu_y] ds + \\ &+ \int_{BC} \left[\left(Mu + \frac{m}{2} u\right)(-(-y)^m u_x, u_y)\right] \cdot \nu ds. \end{aligned} \tag{13}$$

Let us assume that there exists $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ a nontrivial solution of (2) - (3). In the pure power case when we have $F(u) = |u|^p/p$, we could calculate the expression in the integral over Ω on the right-hand side of (13) to be

$$\frac{m}{2} uF'(u) - (m+4)F(u) = \left(\frac{m}{2} - \frac{m+4}{p}\right)|u|^p = \frac{m[p - 2^*(1,m)]}{2p}|u|^p. \tag{14}$$

Thus combining (11), (12) and (14) we obtain

$$\begin{aligned} 0 > \frac{m[2^*(1,m) - p]}{2p} \int_\Omega |u|^p dxdy = \int_{AC} \frac{1}{2} u_\nu^2 [-(-y)^m \nu_x^2 + \nu_y^2][(m+2)x\nu_x + 2y\nu_y] ds + \\ + \int_{BC} \left[Mu + \frac{m}{2} u\right] (-(-y)^m u_x, u_y) \cdot \nu \, ds \geq 0 \end{aligned}$$

which contradicts u being nontrivial.

In the critical case $p = 2^*(1,m)$ the area integral on the right-hand side of (13) vanishes. Then, for the solution $u(x,y)$ of equation (2) the integrals over AC and BC must also vanish. We begin by showing that u has zero trace on BC by applying the Hardy-Sobolev inequality with remainder term (7). Referring to (11), we have

$$\begin{aligned} &\int_{BC} \left[Mu + \frac{m}{2} u\right] (-(-y)^m u_x, u_y) \cdot \nu \, ds = \\ &= \int_{y_C}^0 \left[4(-t)^{\frac{m+2}{2}} w'(t)^2 - \frac{m^2}{4}(-t)^{\frac{m-2}{2}} w(t)^2\right] dt \geq \frac{16}{y_C^2} \int_{y_C}^0 (-t)^{\frac{m+2}{2}} w(t)^2 dt \geq 0 \end{aligned}$$

where $w(t) = u|_{BC}(t) = u(\beta(t))$. Since the integral on BC must vanish we obtain

$$\int_{y_C}^0 (-t)^{\frac{m+2}{2}} w(t)^2 dt = 0$$

from where it follows that $u|_{BC} = 0$.

In the critical case we have already additionally assumed that the noncharacteristic part of AC is strictly *M-star-like*, then (12) shows that

$$u_\nu^2 [-(-y)^m \nu_x^2 + \nu_y^2] = 0 \; on \; AC \tag{15}$$

which implies that the normal derivative u_ν vanishes on the parts of the boundary which are strictly sub-characteristic. This, combined with the boundary condition $u = 0$ on AC yields $u \equiv 0$ on $\overline{\Omega}$. In fact, using the multiplier $Vu = u_y$ one obtains

$$\begin{aligned}\int_\Omega Vu(Lu + F'(u))dxdy &= \frac{m}{2}\int_\Omega (-y)^{m-1}u_x^2 dxdy + \\ &+ \int_{\partial\Omega}\left(-(-y)^m u_x u_y \nu_x + \frac{1}{2}(u_y^2 + (-y)^m u_x^2)\nu_y + F(u)\nu_y\right)ds.\end{aligned} \tag{16}$$

Using the boundary conditions, the fact that $u|_{BC} = 0$ and $F(0) = 0$, (16) becomes

$$\int_{AC\cup BC}\frac{1}{2}\nu_y u_\nu^2(-(-y)^m\nu_x^2 + \nu_y^2)ds + \int_{AB}F(u)ds = -\frac{m}{2}\int_\Omega (-y)^{m-1}u_x^2 dxdy \tag{17}$$

where the first integral vanishes on AC by (15) and by the fact that BC is characteristic of (2) i.e. $-(-y)^m\nu_x^2 + \nu_y^2 = 0$ there. From the fact $F(u) = |u|^p/p \geq 0$, it follows that the left-hand side is non-negative and then, using the right-hand side, thus $u_x = 0$ in Ω. Since $u = 0$ on BC, one concludes $u \equiv 0$ in Ω. $\square$

3. Open problems. We have assumed throughout this work that the solutions are of class $C^2(\overline{\Omega})$ which is clearly too much. In fact, even for the classical Tricomi problems one expects, in general, to have the possibility of isolated singularities in the first derivatives at parabolic boundary points [10]. Because of these facts, it is quite naturally, to study such problems assuming less regularity. Especially, in the frames of generalized solvability, including possible singularities of the solution.

Open Problems:

1. Uniqueness of generalized solution of (2)–(3) in the supercritical or critical case, if AC is not fully characteristic. Actually, the characteristic case is given in [12].

2. The question regarding existence of nontrivial solution of the problem (2)–(3) also was not considered.

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ТЪЖДЕСТВА НА ПОХОЖАЕВ И ПРИЛОЖЕНИЕ КЪМ УРАВНЕНИЯ НА МАТЕМАТИЧЕСКАТА ФИЗИКА ОТ СМЕСЕН ТИП

Недю Попиванов, Яни Бошев

Посвещава се на 80-тата годишнина на проф. А. М. Накушев

За нелинейни уравнения от тип на Гелерщед, за израждаща се хиперболична задача на Коши–Гурса в триъгълна област със субхарактеристика, ще бъде установен резултат за единственост на регулярно решение. За хомогенни *суперкритични* нелинейности, единствеността на нулевото решение в класа на регулярните решения ще бъде установена чрез комбинацията от прилагане на подходящи тъждества на Похожаев, с подходящи предположения за областта и прилагането на неравенството на Харди-Соболев. Ще се направи представяне на някои отворени въпроси, засягащи обобщената разрешимост на задачата.