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TEACHING MATHEMATICAL BEAUTY: INFINITY IN THE ROOM

Latchezar P. Tomov

Mathematical beauty is one of the most important themes in education that can make students curious and intrinsically motivated. One of the most important aspects of beauty is the subject of infinity. In the literature, teaching of infinity is represented with multiple examples, such as progressions, fractals, etc. We present here some other aspects of infinity that can be thought at school – the infinite efficiency of mathematics, the infinite validity of proofs, the relation between infinity and symmetry and the difference between unlimited and infinite. All these examples are used in teaching in NGDEK – The National School for Ancient Languages and Cultures "Constantine-Cyril the Philosopher" with students in 9th grade in 2017–2018.

Introduction. One of the key challenges in mathematics education is development of intrinsic motivation of students. Grades and prizes from contests are form of extrinsic motivation that is as dangerous as it is useful – it conditions student to work for recognition, not for personal pleasure from finding new truths. In the absence of intrinsic motivation, a change in fortune for the student – a temporary shift down of external marks for success can demotivate him or her and further diminish the learning effort. Students with only extrinsic motivation are at constant risk of breaking with mathematics and losing their interest. Mathematical beauty is one of the most important themes in education that can make them curious and self-motivated. One of the most important aspects of beauty is the subject of infinity. Infinity often is the elephant in the room in mathematics teaching in most schools in Bulgaria, neglected or not fully employed as a concept – for fear of discouraging students, or because of the utilitarian concept that school education must only be "practical". School focuses more on solving problems than learning theory with deep understanding. Problem solving is only one part of the set of mathematical skills – great discoveries, as Felix Klein points out, rest in the power of intuition [7].

There are many papers on the topic of teaching infinity in the classroom – [4], [6], [10] and [13], to name a few. They focus on the extension of students' knowledge from the finite into the infinite – for example in arithmetic and geometric progressions. Certain types of discussions are lacking, however – those touching philosophical or metaphysical issues. These are one of the most interesting parts of mathematics and are often topics of popular books like "Levels of infinity" of Hermann Weyl or topics, discussed in philosophy, related to the works of Plato. These discussions can be used to teach the importance of proofs, the beauty of mathematics and – what mathematics really is – the **opposite** of doing unlimited amount of calculations. Everything presented in this paper was tested in 9th grade of NGDEK in 2017–2018.

Infinite efficiency of mathematics. *"Mathematics has been called the science of the infinite. Indeed, the mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite. This is his glory."* [15].

The first thing children learn in school is how to add and extract numbers. Numbers are the key to the explanation of the nature of mathematics. If the teacher asks the question "What is mathematics?" to students, either in 5th grade or in high school, he or she will often get the answer: "Mathematics teaches us how to manipulate numbers". This answer is a good start for further discussion. What are numbers? To what they refer? Here, the teacher can reenact part of the Socratic dialogue by Alfred Renyi [9], to highlight what is the difference between counting sheep, apples and just counting. Here is the first important lesson about the infinite efficiency of mathematics – $1 + 1 = 2$ is a shape that can be applied to **infinitely many** particular situations. It's validity is independent of the nature of objects that we add as long as they are the same type – 1 apple + 1 apple equals 2 apples, but 1 apple and 1 planet equals 2 **objects**. We can use simple arithmetic to explain type inheritance with or without explicit notion of the term – 1 apple + 2 oranges equals 3 fruits – a common **higher type** for both apples and oranges.

Simple, basic arithmetic is pure mathematics, as Hardy points [5]. Numbers are shapes, ideas – the concept that inspired Platonic philosophy. This is particularly useful concept when teaching in a classical school like NGDEK – National School for Ancient Languages and Cultures "Constantine-Cyril the Philosopher". It helps to integrate mathematics with philosophy and history since students there study Greek philosophy extensively, including in ancient Greek language. It is important to show with many examples, from simple to complex – from arithmetic to number theory, how the abstract nature of mathematics encapsulates infinity of applications.

Important lesson to transfer is that numbers like 1, 2, or 5^3 are not less general than variables like x, y, z [5]. The fear of the abstract in mathematics can be overcome, by showing that the meaning of numbers depends on the numeral system. A good exercise is to do comparison between decimal and hexadecimal numeral systems. It can show that the value of the symbols, called digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ relies on interpretation when grouped into sequences. For example, 12 can mean 10+2 or 16+2. For students in classical high school this can be lesson in history of mathematics – with Babylonian numerals (Fig. 1).

The infinite efficiency of mathematics lies in its abstractness. Precisely because it is not science of counting and enumerating **something**, but for counting and enumerating **anything**, it has infinite power. Once learned, the rules of arithmetic are applied repeatedly in different areas of science and arts. This is the first level of infinite efficiency.

The next level of efficiency is the level of formulas. The same way one particular number can capture infinity of applications, one formula can do the same for infinity of numbers. A particularly good teaching example is with triangular numbers (Fig. 2), rectangular numbers, etc. known from the time of Pythagoreans [14].

They have a number of advantages – easy to visualize, related to history, music and culture, they have recurrent relationship and with few simple examples, students can discover a general formula.

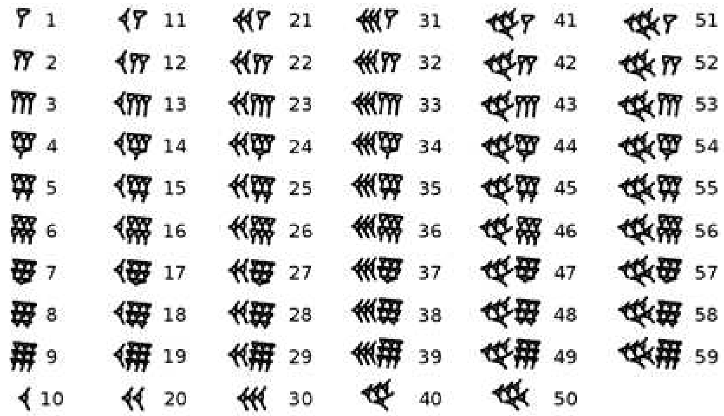

Fig. 1. Babylonian numerals – positional sexagesimal system. Source – Wikipedia

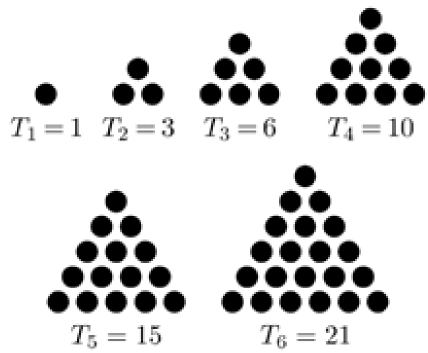

Fig. 2. The first six triangular numbers. Source – Wikipedia

Here each triangular number is related to the previous one:

(1) $T_2 = T_1 + 2, \quad T_3 = T_2 + 3, \quad T_4 = T_3 + 3, \quad T_5 = T_4 + 5, \quad T_6 = T_5 + 6$

(2) $$T_n = T_{n-1} + n$$

The transition between (1) and (2) can be from easy to very hard depending on the level of mathematical skills of the students but it can be done with enough examples. An explanation of the role of this formula should follow, to emphasize its validity for the entire set of natural numbers. A natural follow up is to transform the recursive formula to express the current number from the first one:

(3) $$T_n = T_{n-1} + n = T_{n-2} + n - 1 + n = T_{n-3} + n - 2 + n - 1 + n$$

(4) $$T_n = T_1 + 2 + 3 + \cdots + n = 1 + 2 + 3 + \cdots + n$$

This should be demonstrated first with specific examples, like (1) and then converted to formula as the process of analogy and induction is from specific to general when children first learn how to grasp the abstract. Polya [8] describes a dynamic process from the particular to the general and from the general to the particular with the Pythagorean theorem, which is easier to understand when applied to geometry, where intuition plays significant role. The specific example can be written in two different forms – more "natural" and more resembling (3)–(6).

(5) $T_6 = T_5+6 = T_4+5+6 = T_3+4+5+6 = T_2+3+4+5+6 = T_1+2+3+4+5+6$

(6) $$T_6 = T_{6-1} + 6 = T_{6-2} + 6 - 1 + 6 = T_{6-3} + 6 - 2 + 6 - 1 + 6$$

Once the general formula (4) is established, the teacher can return to specific examples in order to induce hypothesis for shorter version of that calculation. Several different examples with large enough n should convince students how inefficient this procedure is. That is the right moment to lead them to the general formula by examples:

(7) $(1+2+3+4+5) + (5+4+3+2+1) = (6+6+6+6+6) = 5.6 = 5\,(5+1)$

(8) $$(1+2+3+4+5) = \frac{5\,(5+1)}{2}$$

There are several transitions of state in that derivation – from the idea to take the reverse of the sequence, to finding that the sum of the two sequences is the sequence of equal numbers, to know why that happens is one third of the process. The next important step is intuitive – converting summation into product, while the transition $5.6 = 5\,(5+1)$ is not. In practice, the teacher will need several different examples in order to become clear to students so they can discover the formula. Similar approach with geometric examples has been done in [2]. The last step is trivial if they know how to handle fractions.

After student derive the formula, it is the right moment to discuss the next level of infinite efficiency of mathematics. One can quote William Blake from "Auguries of Innocence" [1]:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour

Each number represents an infinity of values depending on possible interpretations. Each equation involving specific numbers, like (8), is applicable to infinity of practical occasions – from the number of chess parties played by 6 people if they play 1 party per opponent, to the number of straight lines passing through 6 points where no 3 points lie on one straight line. The general formula (9) captures an infinity of particular occasions and saves an **unlimited** amount of time and effort in calculation.

(9) $$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}$$

Infinite validity of proofs. Once formulas like (9) are established, it is important to introduce the concept of proof. Proof is the demarcation line between mathematics and all other sciences. There are three particularly important types of proof, suitable for high school or even earlier – **mathematical induction**, **proof by contradiction** and the **method of infinite descent**. The first and the third one are closely related to recursion, and therefore they can help for the integration of mathematics and informatics in school. The third one is more easily understood and more pleasurable for students who like to solve logical problems. It is the crown jewel according to Hardy [5], who gives two examples of *reduction ad absurdum* – both simple, but with vast consequences in development in mathematics. The first one is proof that prime numbers are infinitely many, and the second one is the proof that the square root of two cannot be represented as proportion of two natural numbers (incommensurability). Both are part of the legacy of ancient Greek mathematics from the late Pythagorean School [14], which also makes them more interesting for the students in NGDEK. We used it in a special lecture on applications of Pythagorean Theorem for students in 9th grade at NGDEK that was later published as paper [11].

Before teaching different kinds of proofs, we must clarify why proof is necessary. We can start with Felix Klein's statement [7]:

"Mathematics in general is fundamentally the science of self-evident things."

To many of them this humorous statement causes smiles or laugh. An example from George Polya [8] can explain the seriousness of this statement and the fundamental difference between mathematics and other sciences. For a suitable audience, that has already been introduced to probability theory, we can show the application of Laplace principle of indifference when calculating the probability that the Sun will rise tomorrow, given n known previous days in which it rose. The process of calculation needs to be omitted as it uses higher mathematics, but the result is surprisingly simple:

$$P\left(n|n+1\right) = \frac{n+1}{n+2} \tag{10}$$

The analysis of this probability gives the impression that when the body of data grows with n the confidence in tomorrow approaches 100%. It is a good example of limit and can be used for that purpose, since its examples are simple fractions. This is particular case of another equation, however:

$$P\left(n|n+m\right) = \frac{n+1}{n+m+1} \tag{11}$$

If we vary m we can see what happens when $m \to \infty$, $P\left(n|n+m\right) \to 0$. This can be shown with two different levels of examples – with particular values of n, like 10, 1000, 1000000, and for the general case, when m is multiple of n

$$P\left(n|n+k.n\right) = \frac{n+1}{n+k.n+1} \approx \frac{n}{n+k.n} = \frac{n}{n\left(1+k\right)} = \frac{1}{\left(1+k\right)} \tag{12}$$

It is easy to see that even if we are very confident in the next sunrise, the confidence diminishes with the amount of consecutive days in the future for which we expect this event to repeat. After giving these examples, a relevant conclusion can be drawn by the students, with the help of the teacher – when we have finite amount of data supporting an inductive hypothesis, the probability of this hypothesis being true is zero.

This is a problem of inductive reasoning, the answer to which are proofs. Only proved statements can be considered "truth". A proof of (9) by mathematical induction, which can be written in two lines, encapsulates an infinite chain of proofs $n \to n+1$, which applies for all natural numbers, instantaneously, far more efficient than any computer program. Truth lies beyond the realm of time and the only reachable truth for us is what we know with certainty. That is what drove G. Hardy to career as a mathematician. It was the permanence of results and the eternal glory of the author, surpassing, according to him, that of the great literary writers and poets [5].

Infinity and symmetry. Symmetry is foundational subject in mathematics, from the early development in Greece. The most important is the influence of Plato as a philosopher and as a chair of his Academy in which many mathematicians, like Teetet and Eudoxus, worked. Plato considered the circle and the sphere as the expressions of sublime beauty because of their symmetry [14]. The mark for infinity since Aristotle was a circle. The reason for that is not that motion along the perimeter of the circle can continue indefinitely, but because the order of its symmetry is **infinite**. The perimeter of the circle is a finite number, but the motion along the circle contour is limitless.

Symmetry can be taught in school as a general concept that increases understanding of Euclidean geometry by simplifying concepts and proofs – for example, the proof that the angles at the base of an isosceles triangle are equal, using symmetry, is much simpler to grasp than the "*pons asinorum*" in Elements. The axis of the symmetry is the height from the apex to the base. One can imagine that the triangle is "copied" and reflected along this axis after which it will be congruent to its original (with three equal corresponding sides) from which the equality of the angles will follow. This is a more general and abstract idea. It is simpler than the full and exhaustive proof from Elements, although Charles Dodgson pointed that it only **appears** simpler and will not be shorter if written accordingly with all steps and details. Symmetry for him represents an improvement in **aesthetics**, but not in logic

"Euc. That is, in fact, the same thing as conceiving that there are two coincident Triangles, and that one of them is taken up, turned over, and laid down upon the other. And what does their subsequent coincidence prove? Merely this: that the right-hand angle of the first is equal to the left-hand angle of the second, and vice versa. To make the proof complete, it is necessary to point out that, owing to the original coincidence of the Triangles, this same 'left-hand angle of the second' is also equal to the left-hand angle of the first: and then, and not till then, we may conclude that the base-angles of the first Triangle are equal. This is the full argument, strictly drawn out. The Modern books on Geometry often attain their much-vaunted brevity by the dangerous process of omitting links in the chain; and some of the new proofs, which at first sight seem to be shorter than mine, are really longer when fully stated. In this particular case I think you will allow that I had good reason for not adopting the method of superposition?"[3]

The value of symmetry is not only to be a link between infinity and beauty, although this alone is important for intrinsic motivation of students, particularly people drawn to arts, humanities and philosophy like the students at NGDEK. The value of symmetry is in giving **deep insights** into mathematics. A special and important example is the isoperimetric problem: to determine the plane figure of the largest possible area whose boundary has a specified length. G. Polya [8] gives that problem with some clues to the reader, but it should be done in detail when teaching in the classroom. First, the teacher can analyze the isosceles and equilateral triangles:

$$\frac{S_{eq}}{P_{eq}} = \frac{\frac{\sqrt{3}}{4}a^2}{3a} = \frac{\sqrt{3}a}{12} \tag{13}$$

$$\frac{S_{iso}}{P_{iso}} = \frac{\frac{b}{4}\sqrt{4a^2 - b^2}}{2a + b} \tag{14}$$

Here, formula (13) is a special case of (14) for $b = a$. When $b < a$ the order of symmetry changes from 3 to 2 and with decreasing b while keeping $P_{iso} = 2a + b$ fixed we can see how the ratio of area to perimeter diminishes to zero. A similar and simpler example is with the square and the rectangle with area of 1 square unit. For $b = \frac{1}{a}$ the rectangle will have constant area and variable perimeter which can increase arbitrarily – $P_{req} \to \infty$ with $a \to \infty$ while the area remains constant. Thus, the ratio of area and perimeter approaches zero with the increase of the deviation from the square, which has higher order of symmetry. A further and more complex comparison is the one between different regular polygons and the increase of the ratio of area vs perimeter with the increase of the order of symmetry – from triangle, square, to pentagon, hexagon, etc. For a more advanced group of students the teacher can show the method of Archimedes for approximating π with inscribed and circumscribed regular polygons. Students can calculate the ratio of area to perimeter at every step of the procedure and see how it increases with the number of sides of the polygons, approaching the case for the circle.

Conclusion. Infinite efficiency of mathematics lies in three things:

(1) the ability to represent an infinite variety of different specific calculations in finite amount of symbols

(2) the ability to do infinite amount of calculations in finite time – the series summations, for example

(3) the infinite validity of proof and the ability to have certain results, independent of time.

The beauty of mathematics is mirror reflection of its efficiency. There is plenty of room for infinity, for it can occupy arbitrarily small space and time, just as well as it occupies arbitrarily large quantities.

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Latchezar P. Tomov
New Bulgarian University
Computer Science Department
21, Montevideo Str.
1618 Sofia, Bulgaria
e-mail: luchesart@gmail.com

ПРЕПОДАВАНЕ НА КРАСОТАТА НА МАТЕМАТИКАТА: БЕЗКРАЙНОСТ В СТАЯТА

Лъчезар Томов

Красотата на математиката е една от най-важните теми в образованието, която може да събуди любопитството на студентите и да развие вътрешнтата им мотивация. Един от най-важните аспекти на тази красота е безкрайността. В литературата преподаването на безкрайността е представено с множество примери, като прогресии, фрактали и др. Тук предлагаме някои други аспекти на безкрайността, които могат да се преподават в училище – безкрайната ефикасност на математиката, безкрайната валидност на доказателствата, връзката между безкрайност и симетрия и разликата между неограниченото и безкрайното. Всички тези пример са използвани в преподаването на математика в НГДЕК – Национална гимназия за древни езици и култури „Константин-Кирил Философ“ в 9-ти клас през учебната 2017–2018-а година.