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UNIVERSAL BOUNDS FOR CARDINALITIES AND ENERGY OF CODES IN HAMMING SPACES

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We survey recent results on universal bounds on the energy of codes in Hamming spaces. The universality means, in particular, that the bounds hold for a large class of potential functions (the most important bounds – for absolutely monotone interactions). Furthermore, we employ signed measures that are positive definite up to certain degrees to establish Levenshtein-type upper bounds on the cardinality of codes with given minimum and maximum distance, and universal lower bounds on the potential energy for codes with given maximum distance and cardinality. In particular, the results extend the Levenshtein framework.

1. Introduction. Let H_q^n be the Hamming space over $H_q = \{0, 1, \ldots, q-1\}$, where q is not necessarily a prime power, i.e. we do not need any structure of the alphabet. The Hamming distance $d(x, y)$ between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ from H_q^n the usual one, i.e. $d(x, y)$ is equal to the number of coordinates in which they differ.

We refer to any non-empty set $\mathcal{C} \subset H_q^n$ as a *code*. The main parameters of a code $\mathcal{C}$ are the dimension n, the cardinality $|\mathcal{C}| = M$ and the minimum distance $d = d(\mathcal{C}) := \min\{d(x, y) \, : \, x, y \in \mathcal{C}, x \neq y\})$. A set $\mathcal{C}$ with such parameters will be referred to as $(n, M, d)_q$-code.

In this paper we describe some recent results of investigations of codes in Hamming spaces. The problems of obtaining the maximum possible cardinality of $(n, M, d)_q$-codes and the smallest possible potential energy of such codes are subject of investigations of many researchers [11, 13, 15, 18, 19, 20]. The main tools used to obtained the presented results are the linear programming techniques. First, we use the theory of orthogonal polynomials which are orthogonal with respect to classical positive measures. Next, we develop and apply the necessary theory of signed measures to derived the Levenshtein-type upper bounds on the cardinality of codes with given minimum and maximum distance.

The paper is organized as follows. In Section 2 we collect the main notions and introduce the Krawtchouk polynomials and their adjacent polynomials. Section 3 is devoted to universal bounds for the size of codes and designs in Hamming spaces. In Section

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4 universal lower bounds on the potential energy for codes are shown. These bounds are obtained by Boyvalenkov, Dragnev, Hardin, Saff and the author [8, 9]. Employing signed measures that are positive definite up to certain degrees, the Levenshtein-type upper bounds on the cardinality of codes with given minimum and maximum distance are obtained. In Section 5 test-functions are introduced. They give necessary and sufficient conditions for existence of improving polynomials of higher degrees for obtaining the so-called "second level" bounds.

2. Linear programming framework in H_q^n.

2.1. Codes and designs in H_q^n and their energy. Let $\mathcal{C} \subset H_q^n$ be a $(n, M, d)_q$-code. For the linear programming techniques it is convenient to work with the "inner product"

$$\langle x, y\rangle := 1 - \frac{2d(x,y)}{n} = t_{n-d} \in T_n,$$

instead of the distance $d(x, y)$ (see [19]), where we denote by $Z_n := \{0, 1, \ldots, n\}$ and $T_n := \left\{t_i = -1 + \frac{2i}{n} \,:\, i = 0, 1, \ldots, n\right\}$ the sets of all possible distances and the set of all possible inner products between points in H_q^n, respectively. In addition to the discrete sets $T_n \subset [-1, 1]$ and $Z_n \subset [0, n]$ we shall use the complete intervals $0 \le z \le n$ and $-1 \le t \le 1$ with dependent variables z and t related through the equation $t = 1 - \frac{2z}{n}$.

Then the minimum distance $d(\mathcal{C})$ corresponds to the maximal inner product

$$s = s(\mathcal{C}) = 1 - \frac{2d(\mathcal{C})}{n} := \max\{\langle x, y\rangle \,:\, x, y \in \mathcal{C}, x \neq y\} \in T_n,$$

and we will consider $\mathcal{C}$ to be $(n, M, s)_q$-code too. We also use below the minimal inner product

$$\ell = \ell(\mathcal{C}) := \min\{\langle x, y\rangle \,:\, x, y \in C, x \neq y\} \in T_n,$$

corresponding to the maximum distance $d_{\max} = d_{\max}(\mathcal{C}) = n(1-\ell)/2$ of a code $\mathcal{C}$.

For fixed n, q and s, let

$$\mathcal{A}_q(n, s) := \max\{|\mathcal{C}| \,:\, \mathcal{C} \text{ be a spherical } (n, M, s)_q\text{-code}\}$$

be the maximum possible cardinality of a $(n, M, s)_q$-code $\mathcal{C}$.

One of the classical problems in coding theory is the investigation of the quantity $\mathcal{A}_q(n, s)$. The linear programming techniques as explained below are a powerful tool for obtaining the universal bounds on $\mathcal{A}_q(n, s)$ and their refinements.

For a given potential function $h : [-1, 1) \to (0, +\infty)$, we define the h-energy of a code $\mathcal{C} \in H_q^n$ by

$$E_h(n, \mathcal{C}) := \frac{1}{|\mathcal{C}|} \sum_{x,y \in \mathcal{C}, x \neq y} h(\langle x, y\rangle).$$

Many problems of interest (cf. [2, 3, 4, 11, 21]) can be formulated as minimizing the quantity $E_h(n, \mathcal{C})$ for a suitable h over codes $\mathcal{C}$ of fixed cardinality; that is, to determine

$$\mathcal{E}_h(n, M) := \min\{E_h(n, \mathcal{C}) \,:\, |\mathcal{C}| = M\},$$

the minimal h-energy of a code $\mathcal{C} \subset H_q^n$ of cardinality M. While we only need the values of h on the discrete set T_n for computing the h-energy, we shall further assume that h is strictly/absolutely monotone on the interval [-1,1); that is, h and all its derivatives are defined and positive/non-negative on this interval. We remark that $F(z) = h(t)$, where $z = n(1-t)/2$, is completely monotone on $(0, n]$ (that is, $(-1)^k F^{(k)}(z) \geq 0$ for all $z \in (0, n]$) if and only if h is absolutely monotone on $[-1, 1]$.

Characterization of codes according to their strength as a design (the exact definition is presented below) started with Delsarte [12], where $\tau + 1 = d'$ is the dual distance of the code C (see also [13, 15, 18, 19]).

For fixed strength τ and dimension n let

$$B(n, \tau) = \min\{|\mathcal{C}| \ : \ \exists \ \tau\text{-design } \mathcal{C} \subset H_q^n\}.$$

Since the designs are, in a certain sense, an approximation of the whole space H_q^n, they play an important role in the investigations of energy problems of such codes. Energies of codes and designs (or orthogonal arrays) in H_q^n were considered first in 2014 by Cohn and Zhao [11].

Furthermore, let

$$\mathcal{C}_{n,q}(\ell, s) := \left\{\mathcal{C} \subset H_q^n \ : \ \ell \leq \ell(\mathcal{C}), \ s(\mathcal{C}) \leq s\right\}$$

be the set of codes in H_q^n with pairwise distances greater than or equal to the minimum distance d and less than or equal to the maximum distance D. Let

$$\mathcal{A}_q(n, \ell, s) := \max\left\{|\mathcal{C}| \ : \ \mathcal{C} \in \mathcal{C}_{n,q}(\ell, s)\right\}$$

be the maximum possible cardinality of a code from $\mathcal{C}_{n,q}(\ell, s)$.

For absolutely monotone potentials h we consider the quantity

$$\mathcal{E}_h(n, M, \ell) := \min\{E_h(\mathcal{C}) \ : \ \mathcal{C} \in \mathcal{C}_{n,q}(\ell, 1 - 2/n), \ |\mathcal{C}| = M\},$$

the smallest possible h-energy of a code from $\mathcal{C}_{n,q}(\ell, 1 - 2/n)$ with prescribed M.

The uniquely expansion of any real polynomial in terms of the Krawtchouk polynomials plays an essential role in the techniques of linear programming.

2.2. Krawtchouk polynomials and their adjacent polynomials. For fixed n and q, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n,q)}(t) := \frac{1}{r_i} K_i^{(n,q)}(z),$$

where $z = \dfrac{n(1-t)}{2}$, $r_i = r_i^{(n,q)} := \dbinom{n}{i}(q-1)^i$, and

$$K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{z}{j} \binom{n-z}{i-j}, \quad i = 0, 1, \ldots, n,$$

are the (usual) Krawtchouk polynomials corresponding to H_q^n (see [1, 23, 12]).

The polynomials $K_i^{(n,q)}(z)$ can be defined by the following three-term recurrence relation

$$(i+1) K_{i+1}^{(n,q)}(z) = [i + (q-1)(n-i) - qz] K_i^{(n,q)}(z) - (q-1)(n-i+1) K_{i-1}^{(n,q)}(z),$$

for $1 \leq i \leq n-1$, where $K_0^{(n,q)}(z) = 1$ and $K_1^{(n,q)}(z) = n(q-1) - qz$.

The measure of orthogonality [14] for the system $\{Q_i^{(n,q)}(t)\}_{i=0}^n$ is a discrete measure given by

$$d\mu_n(t) := q^{-n} \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \delta_{t_i},$$

where δ_{t_i} is the Dirac-delta measure at $t_i \in T_n$. Note that the form

$$\langle f, g \rangle = \int f(t)g(t)d\mu_n(t)$$

defines an inner product over the class $\mathcal{P}_n$ of polynomials of degree less than or equal to n.

We also need the so-called *adjacent polynomials* as introduced by Levenshtein (cf. [19, Section 6.2], see also [16, 17, 18])

$$\begin{aligned} Q_i^{(1,0)}(t) &= Q_i^{(1,0,n,q)}(t) = \frac{K_i^{(n-1,q)}(z-1)}{\sum_{j=0}^{i} \binom{n}{j}(q-1)^j}, \\ Q_i^{(1,1)}(t) &= Q_i^{(1,1,n,q)}(t) = \frac{K_i^{(n-2,q)}(z-1)}{\sum_{j=0}^{i} \binom{n-1}{j}(q-1)^j}, \\ Q_i^{(0,1)}(t) &= Q_i^{(0,1,n,q)}(t) = \frac{K_i^{(n-1,q)}(z)}{\binom{n-1}{i}(q-1)^i}, \end{aligned}$$

where $z = n(1-t)/2$. The corresponding measures of orthogonality are, respectively,

$$(1-t)d\mu_n(t), \quad (1-t)(1+t)d\mu_n(t), \quad (1+t)d\mu_n(t).$$

If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree $m \leq n$, then $f(t)$ can be uniquely expanded in terms of the Krawtchouk polynomials as $f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t)$. Note that if the degree of the polynomial $f(t)$ exceeds n, then $f(t)$ is taken modulo $\prod_{i=0}^{n}(t - t_i)$.

3. Universal bounds for the size of codes and designs in Hamming spaces. The linear programming method was introduced in coding theory by Delsarte [12] and developed later by many researchers (see, for example [13, 19] and references therein).

3.1. General linear programming bounds. Let

$$F_{\geq} := \{f(t) \,:\, f_0 > 0, f_i \geq 0, i = 1, 2, \ldots, n\}.$$

If $f_i > 0$ for $i = 0, 1, 2, \ldots, \deg(f)$, then we write $f(t) \in F_{>}$.

The next theorem gives the linear programming upper bound on $\mathcal{A}_q(n, s)$.

Theorem 3.1 (LP bounds for codes in H_q^n, [12])**.** *Let n, q, and s be fixed. Then*

$$\mathcal{A}_q(n, s) \leq \min_{f \in \mathcal{F}_{n,s}} \frac{f(1)}{f_0},$$

where

$$\mathcal{F}_{n,s} := \{f \in F_{\geq} \,:\, f(t) \leq 0,\, t \in [-1, s]\}.$$

Similarly, we have the following linear programming upper bound on $\mathcal{A}_q(n, \ell, s)$.

Theorem 3.2 (LP bounds for codes in $\mathcal{C}_{n,q}(\ell, s)$, [12])**.** *Let n, q, ℓ, and s be fixed. Then*

$$\mathcal{A}_q(n, \ell, s) \leq \min_{f \in \mathcal{F}_{n,\ell,s}} \frac{f(1)}{f_0},$$

where

$$\mathcal{F}_{n,\ell,s} := \{f \in F_{\geq} \,:\, f(t) \leq 0,\, t \in [\ell, s]\}.$$

Let us set the following two subsets of the class $\mathcal{P}_n$:

$$\mathcal{A}_{n,M;h} := \{f(t) \in F_{\geq} \, : \, f(t) \leq h(t) \text{ for every } t \in T_n\},$$

and

$$\mathcal{A}_{n,\ell;h} := \{f(t) \in F_{\geq} \, : \, f(t) \leq h(t), \; t \in [\ell, 1)\}.$$

There are known lower bounds analogous for Hamming spaces to Yudin's linear programming lower bounds for the energy of spherical codes [24], see also [11, Proposition 5].

Theorem 3.3 (LP bounds for energy of the codes in H_q^n, [24])**.** *Let n be a positive integer and h be an absolutely monotone function on $[-1, 1)$. Then*

$$\mathcal{E}_h(n, M) \geq \max_{f \in \mathcal{A}_{n,M;h}} (f_0 M - f(1)), \qquad \textit{for every } M \geq 2.$$

Theorem 3.4 (LP bounds for energy of the codes in $\mathcal{C}_{n,q}(\ell, s)$, [24])**.** *Let n be a positive integer, h be an absolutely monotone function on $[-1, 1)$, and ℓ be a fixed minimal inner product. Then*

$$\mathcal{E}_h(n, M, \ell) \geq \max_{f \in \mathcal{A}_{n,\ell;h}} (f_0 M - f(1)).$$

3.2. Levenshtein bounds. Here we explain the Levenshtein universal upper bounds on the maximum possible cardinality of a $(n, M, s)_q$-code, i.e. on quantity $A_q(n, s)$. It is important to note that, in Section 4 (see Theorem 4.6 below) the explicit upper bounds for $\mathcal{A}_q(n, \ell, s)$ are obtained. These bounds can be computed for all feasible values of q, n, s, and ℓ, which makes them universal in the sense of Levenshtein bounds [19].

For $a, b \in \{0, 1\}$ and $i \in \{1, 2, \ldots, n - a - b\}$, denote by $t_i^{a,b}$ the greatest zero of the adjacent polynomial $Q_i^{(a,b)}(t)$ and also define $t_0^{1,1} = -1$. We have the interlacing properties $t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1}$, see [19, Lemmas 5.29, 5.30]. For a positive integer τ, let $\mathcal{I}_\tau$ denote the interval

$$\mathcal{I}_\tau := \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0}\right], & \text{if } \tau = 2k - 1, \\ \left[t_k^{1,0}, t_k^{1,1}\right], & \text{if } \tau = 2k. \end{cases}$$

Then the intervals $\mathcal{I}_\tau$ are well defined and form partition $\mathcal{I} = [-1, 1)$ into subintervals with non-overlapping interiors.

Using suitable polynomials Levenshtein obtained the following universal upper bounds for $A_q(n, s)$.

Theorem 3.5 (Levenshtein bound, [19, Equation (6.45) and (6.46)])**.** *For every τ and $s \in \mathcal{I}_\tau$, the Levenshtein bound*

$$A_q(n,s) \leq \begin{cases} L_{2k-1}(n,s) = \left(1 - \dfrac{Q_{k-1}^{(1,0)}(s)}{Q_k^{(n,q)}(s)}\right) \displaystyle\sum_{j=0}^{k-1} \binom{n}{j} (q-1)^j, & \textit{if } \tau = 2k-1, \\ L_{2k}(n,s) = q\left(1 - \dfrac{Q_{k-1}^{(1,1)}(s)}{Q_k^{(0,1)}(s)}\right) \displaystyle\sum_{j=0}^{k-1} \binom{n-1}{j} (q-1)^j, & \textit{if } \tau = 2k. \end{cases} \tag{1}$$

holds.

As mentioned above, some characteristics of the designs in H_q^n play an important role in the investigations of energy problems of codes in Hamming spaces. There are several equivalent definitions of designs and here we present the one useful in obtaining bounds on $B(n,\tau)$.

Definition 3.6. *A τ-design $\mathcal{C} \subset H_q^n$ is a code such that the equality*

$$\sum_{x\in\mathcal{C}} f(\langle x,y\rangle) = f_0|\mathcal{C}|$$

holds for any point $y \in H_q^n$ and any real polynomial $f(t) = \sum_{i=0}^{r} f_i Q_i^{(n,q)}(t)$ of degree $r \le \tau$.

Remark 3.7. In statistics and combinatorics, the designs in Hamming spaces are known as *orthogonal arrays* and have been studied for a wide range of practical applications such as planning experiments, parts testing, drug testing, clinical trials, and also in computer science for software testing, big data, and data protection. There are also applications of orthogonal arrays in cryptography. The main reference for orthogonal arrays is the book "Orthogonal Arrays: Theory and Applications" by Hedayat, Sloane and Stuffken [15].

The classical universal lower bound on $B(n,\tau)$ is due to Rao [22] (see also [13, 15, 19]).

Theorem 3.8 (Rao bound,[22])**.** *Let n, q and τ be fixed. Then*

$$B(n,\tau) \ge R_q(n,\tau) := \begin{cases} q\displaystyle\sum_{i=0}^{k-1}\binom{n-1}{i}(q-1)^i, & \text{if } \tau = 2k-1,\\ \displaystyle\sum_{i=0}^{k}\binom{n}{i}(q-1)^i, & \text{if } \tau = 2k.\end{cases} \tag{2}$$

There are the following important connections between the Rao (2) and the Levenshtein (1) bounds.

Theorem 3.9. *The bounds* (1) *and* (2) *are related by the equalities*

$$\begin{aligned} L_{2k-2}(n,t_{k-1}^{1,1}) = L_{2k-1}(n,t_{k-1}^{1,1}) = R_q(n,2k-1),\\ L_{2k-1}(n,t_k^{1,0}) = L_{2k}(n,t_k^{1,0}) = R_q(n,2k)\end{aligned} \tag{3}$$

at the ends of the intervals $\mathcal{I}_\tau$.

The relations (3) in Theorem 3.9 explain and justify the connection between the cardinality M and the strength (degree) $\tau = \tau(n,M)$. In fact, for fixed cardinality M, Rao bounds show which parameters must be chosen in order to obtain universal lower bounds on $\mathcal{E}_h(n,M)$ and $\mathcal{E}_h(n,M,\ell)$. More precisely, for given length n and cardinality M, we find the unique

$$\tau := \tau(n,M) \ \text{ such that } \ M \in (R_q(n,\tau), R_q(n,\tau+1)]\,.$$

Then all other necessary parameters come with n, M and τ as shown in the next subsection.

3.3. Levenshtein-type quadrature. Next we explain the useful quadrature that can be applied for deriving and calculation of universal lower bounds on energies of the codes and designs in Hamming spaces.

Levenshtein [17] proves (see also [19, Section 5, Theorem 5.39] and [18]) that for every fixed (cardinality) $M > R_q(n,2k-1)$ there exist uniquely determined real numbers (called nodes) $-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and positive numbers (called their weights) $\rho_0, \rho_1, \dots, \rho_{k-1}$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

holds for every real polynomial $f(t)$ of degree at most $2k-1$.

The numbers α_i, $i = 0, 1, \dots, k-1$, are the roots of the equation

$$Q_k^{(1,0)}(t) Q_{k-1}^{(1,0)}(\alpha_{k-1}) - Q_k^{(1,0)}(\alpha_{k-1}) Q_{k-1}^{(1,0)}(t) = 0. \quad (4)$$

It is convenient to find first $\alpha_{k-1} = s$ from the equation $M = L_{2k-1}(n,s)$ and to solve then (4).

Similarly, for every fixed number (cardinality or non-integer) $M > R_q(n, 2k)$ there exist uniquely determined real numbers (nodes) $-1 = \beta_0 < \beta_1 < \cdots < \beta_k < 1$ and positive numbers (their weights) $\gamma_0, \gamma_1, \dots, \gamma_k$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k} \gamma_i f(\beta_i)$$

holds for every real polynomial $f(t)$ of degree at most $2k$. The numbers β_i, $i = 1, \dots, k$, are the roots of the equation

$$Q_k^{(1,1)}(t) Q_{k-1}^{(1,1)}(\beta_k) - Q_k^{(1,1)}(\beta_k) Q_{k-1}^{(1,1)}(t) = 0. \quad (5)$$

Similarly to the odd case, $\beta_k = s$ can be found from the equation $M = L_{2k}(n,s)$ and then (5) can be solved.

We also need and use the kernels (see (2.69) in [19, Section 2]; also Section 5 in [19])

$$T_k(u,v) = \sum_{i=0}^{k} r_i Q_i^{(n,q)}(u) Q_i^{(n,q)}(v) = c \cdot \frac{Q_{k+1}^{(n,q)}(u) Q_k^{(n,q)}(v) - Q_{k+1}^{(n,q)}(v) Q_k^{(n,q)}(u)}{u - v}$$

(c is a positive constant, $u \neq v$, this is in fact the Christoffel-Darboux formula). Note that the $(1,0)$ and $(1,1)$, which are analogs of $T_k(u,v)$, define the Levenshtein polynomials.

4. Universal energy bounds for codes in Hamming spaces.

4.1. Energy bounds for $\mathcal{E}_h(n, M)$. Boyvalenkov, Dragnev, Hardin, Saff and the author [8, 9] applied linear programming techniques to derive explicit lower bounds for $\mathcal{E}(n, M; h)$. These bounds can be computed for all feasible values of q, n, s, and ℓ, (respectively) which makes them universal in the sense of Levenshtein [19].

Theorem 4.1 ([8, Theorem 4.2])**.** *If n is a positive integer and h is absolutely monotone on $[-1, 1)$, then*

$$\mathcal{E}(n, M; h) \geq \begin{cases} M \sum_{i=0}^{k-1} \rho_i h(\alpha_i), & \forall M \in (R_q(n, 2k-1), R_q(n, 2k)], \\ M \sum_{i=0}^{k} \gamma_i h(\beta_i), & \forall M \in (R_q(n, 2k), R_q(n, 2k+1)]. \end{cases} \quad (6)$$

It is clear that all maximal codes which attain the Levenshtein bounds $L_\tau(n,s)$, have the necessary strength and the suitable inner products and therefore achieve the bounds (6) as well. In [11] (see also [3]), a table of universally optimal codes in H_q^n is presented. Clearly, all codes which attain (6) are universally optimal (for more details see [8]).

Recall that the Levenshtein bound (see [17, 18, 19]) and the energy bound [8] work for $\ell=-1$. The bounds depend on the properties of Krawtchouk polynomials and their adjacent polynomials which are orthogonal with respect to classical positive measures (discussed at the end of Subsection 2.2). For the case $\ell>-1$, however, we need to develop and apply the necessary theory of signed measures.

4.2. Energy bounds for $\mathcal{E}_h(n,M,\ell)$. For $\ell \in T_n$ we introduce further adjacent polynomials $Q_i^{1,\ell}(t)$ as generalizations of the polynomials $Q_i^{1,1}(t)$ (note that $\ell=-1$ in $Q_i^{1,\ell}(t)$ gives $Q_i^{1,1}(t)$; see definitions in [19, Eqn. (5.66)]).

Suppose that the numbers k, ℓ, and s are chosen to satisfy the following conditions

$$-1 \le \ell < t_{k,1}^{1,0} < t_{k,k}^{1,0} < s \le 1, \tag{7}$$

and

$$\frac{Q_{k+1}^{1,0}(\ell)}{Q_k^{1,0}(\ell)} < 1. \tag{8}$$

Theorem 4.2 ([9]). *For given positive integers $n\ge 2$, $q\ge 2$, let k, ℓ, and s be such that the inequalities* (7) *are satisfied. Then the following two classes of orthogonal polynomials are well-defined:*

$$\{Q_j^{1,\ell}(t)\}_{j=0}^{k},\ \text{w.r.t. } d\mu_{n,\ell}(t),\ Q_j^{1,\ell}(1)=1;$$

$$\{Q_j^{1,\ell,s}(t)\}_{j=0}^{k-1},\ \text{w.r.t. } d\mu_{n,\ell,s}(t),\ Q_j^{1,\ell,s}(1)=1.$$

The Gram-Schmidt orthogonalization procedure implies the existence and uniqueness (for the so-chosen normalizations) of the orthogonal series from Theorem 4.2. More precisely, the polynomials $Q_i^{1,\ell}(t)$ can be defined as follows. Let us assume that

$$t_{k+1,1}^{1,0} < \ell < t_{k,1}^{1,0}. \tag{9}$$

Theorem 4.3 ([9, Theorem 4.2]). *Let n, q, k, and ℓ be such that* (8) *and* (9) *are satisfied. Then*

$$Q_i^{1,\ell}(t) = \frac{T_i^{1,0}(t,\ell)}{T_i^{1,0}(1,\ell)} = \eta_i^{1,\ell}t^i+\cdots,\quad i=0,1,\ldots,k,$$

with $\eta_i^{1,\ell}>0$ and the polynomial $Q_i^{1,\ell}(t)$ has i simple zeros $t_{i,1}^{1,\ell}<t_{i,2}^{1,\ell}<\cdots<t_{i,i}^{1,\ell}$ in the interval $(\ell,1)$. Furthermore, the following interlacing properties

$$t_{i,j}^{1,\ell} \in \left(t_{i,j}^{1,0}, t_{i+1,j+1}^{1,0}\right),\ i=1,\ldots,k-1, j=1,\ldots,i;$$

$$t_{k,j}^{1,\ell} \in \left(t_{k+1,j+1}^{1,0}, t_{k,j+1}^{1,0}\right),\ j=1,\ldots,k-1,$$

and, finally, $t_{k,k}^{1,\ell} \in \left(t_{k+1,k+1}^{1,0},1\right)$ hold true.

The polynomials $Q_i^{1,\ell}(t)$ are orthogonal with respect to the signed measure

$$d\mu_{n,\ell}(t) := c^{1,\ell}(t-\ell)(1-t)d\mu_n(t),$$

where

$$c^{1,\ell} := \frac{nq^2}{2(q-1)(2(n-1)-nq(1+\ell))}$$

is a normalizing positive constant. With the next step, this new series can be used for the construction of polynomials $Q_i^{1,\ell,s}(t)$ which are orthogonal with respect to the signed measure

$$d\mu_{n,\ell,s}(t) := c^{1,\ell,s}(t-\ell)(s-t)(1-t)d\mu_n(t),$$

where

$$c^{1,\ell,s} := \frac{n^2q^3}{2(q-1)[4(n-1)(nq(2+\ell+s)/2-n-q+2)-n^2q^2(1+\ell)(1+s))]}.$$

Consider the Christoffel-Darboux kernel associated with the polynomials $Q_j^{1,\ell}(t)$ (see [9]):

$$R_i^{1,\ell}(x,y) \quad := \quad \sum_{j=0}^{i} r_j^{1,\ell} Q_j^{1,\ell}(x) Q_j^{1,\ell}(y) = r_i^{1,\ell} b_i^{1,\ell} \frac{Q_{i+1}^{1,\ell}(x)Q_i^{1,\ell}(y) - Q_{i+1}^{1,\ell}(y)Q_i^{1,\ell}(x)}{x-y},$$

for $0 \le i \le k-1$ (when $x=y$ appropriate derivatives are used). Given (9) and assuming that $t_{k,k}^{1,0} < s < t_{k,k}^{1,\ell}$ we define

$$Q_i^{1,\ell,s}(t) := \frac{R_i^{1,\ell}(t,s)}{R_i^{1,\ell}(1,s)}, \quad i=0,1,\ldots,k-1.$$

Again the Gram-Schmidt orthogonalization provides the existence and uniqueness (for the normalization from Theorem 4.2) of the polynomials $Q_i^{1,\ell,s}(t)$.

In addition to (8) we require

$$\frac{Q_k^{1,\ell}(s)}{Q_{k-1}^{1,\ell}(s)} > \frac{Q_k^{1,\ell}(\ell)}{Q_{k-1}^{1,\ell}(\ell)} \tag{10}$$

in order to get the smallest root of $Q_{k-1}^{1,\ell,s}(t)$ inside the interval $(\ell, t_{k,1}^{1,\ell})$. Then we can define the Levenstein-type polynomials $f_{2k}^{n,\ell,s}(t)$ as follows:

$$f_{2k}^{n,\ell,s}(t) := (t-\ell)(t-s)\left(Q_{k-1}^{1,\ell,s}(t)\right)^2. \tag{11}$$

They will be applied to derive the universal upper bounds for $\mathcal{A}_q(n,\ell,s)$ and lower bounds for $\mathcal{E}_h(n,M,\ell)$ in an approach which is analogous to the case $\ell=-1$.

As in the classical Levenshtein case with $\ell=-1$, denote the roots (again called nodes) of the polynomial (11) by $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < \alpha_k < 1$ and let $L_i(t)$, $i=0,1,\ldots,k+1$ be the Lagrange basic polynomials generated by these nodes. Then we can define the corresponding positive weights by

$$\rho_i := \int_{-1}^{1} L_i(t)d\mu_n(t), \ i=0,1,\ldots,k+1.$$

Furthermore, the roots of the polynomial $f_{2k}^{n,\ell,s}(t)$ form the Radau quadrature formula with positive weights.

Theorem 4.4. *Let $\alpha_0 := \ell$ and $\alpha_k := s$. Then the Radau quadrature formula*

$$f_0 = \int_{-1}^{1} f(t)d\mu_n(t) = \rho_{k+1}f(1) + \sum_{i=0}^{k} \rho_i f(\alpha_i)$$

is exact for all polynomials of degree at most $2k$. Moreover, the weights ρ_i, $i=0,\ldots,k$, are positive, and $\rho_{k+1} > 0$ provided $(t-\ell)Q_k^{1,\ell}(t) \in F_>$.

In the proof of the positive definiteness of his polynomials Levenshtein used (see [19, (3.88) and (3.92)]) what he called the strengthened Krein condition

$$(t+1)Q_i^{1,1}(t)Q_j^{1,1}(t) \in F_> \tag{12}$$

for every $i, j \in \{0, 1, \ldots, n-3\}$. We need the following modification.

Definition 4.5. *We say that the polynomials $\{Q_i^{1,\ell}(t)\}_{i=0}^{k}$ satisfy (k,ℓ)-strengthened Krein condition if*

$$(t-\ell)Q_i^{1,\ell}(t)Q_j^{1,\ell}(t) \in F_> \tag{13}$$

for every $i, j \in \{0, 1, \ldots, k\}$ except possibly for $i = j = k$.

The strengthened Krein condition (12) holds true in F_q^n for all admissible i and j (see [19, Lemma 3.25]). However, the (k,ℓ)-strengthened Krein condition (13) is not true for every ℓ, and for fixed ℓ it is true only for relatively small k. On the other hand, for fixed n, all relevant pairs (k,ℓ) are finitely many and can be therefore subject to computational checks.

Let

$$S_j = \sum_{i=0}^{j} r_i = \sum_{i=0}^{j} (q-1)^i \binom{n}{i}$$

for $j \in \{k-1, k, k+1\}$.

The Levenshtein-type universal upper bound on $\mathcal{A}_q(n,\ell,s)$ as an analog of Theorem 5.42 of [19] follows.

Theorem 4.6 ([9, Theorem 5.2, part 1])**.** *Let n, q, k, $\ell \in (t_{k+1,1}^{1,0}, t_{k,1}^{1,0})$, and $s \in (t_{k,k}^{1,0}, t_{k,k}^{1,\ell})$ be such that the conditions* (8) *and* (10) *are fulfilled and the (k,ℓ)-strengthened Krein condition is satisfied. Then the polynomial $f_{2k}^{n,\ell,s}(t)$ belongs to $\mathcal{F}_{n,\ell,s}$ and, therefore,*

$$\mathcal{A}_q(n,\ell,s) \le \frac{f_{2k}^{n,\ell,s}(1)}{f_0} = \frac{1}{\rho_{k+1}} := L_{2k}(n,\ell,s), \tag{14}$$

where

$$L_{2k}(n,\ell,s) = \frac{S_k\left(Q_{k-1}^{1,\ell}(s) - Q_k^{1,\ell}(s)\right)}{\frac{r_{k+1}Q_{k+1}(\ell)Q_{k-1}^{1,\ell}(s)}{S_{k+1}\left(Q_{k+1}^{1,0}(\ell)-Q_k^{1,0}(\ell)\right)} - \frac{r_kQ_k(\ell)Q_k^{1,\ell}(s)}{S_{k-1}\left(Q_k^{1,0}(\ell)-Q_{k-1}^{1,0}(\ell)\right)}}.$$

Furthermore, an universal lower bound on $\mathcal{E}_h(n,M,\ell)$, as an analog of the universal lower bound for $\mathcal{E}_h(n,M,-1)$ from [8], follows.

Theorem 4.7 ([9, Theorem 5.2, part 2])**.** *For fixed ℓ, for h being an absolutely monotone function on $[-1,1)$, and for M determined by $f_{2k}^{n,\ell,s}(1) = Mf_0$, the Hermite interpolant*[1]

$$g_{2k}^{n,\ell,M}(t) := H((t-s)f_{2k}^{n,\ell,s}(t); h)$$

belongs to $\mathcal{A}_{n,\ell;h}$, and, therefore,

$$\begin{aligned} \mathcal{E}_h(n,M,\ell) &\ge M(Mg_0 - g_{2k}^{n,\ell,M}(1)) \\ &= M^2\sum_{i=0}^{k}\rho_i h(\alpha_i). \end{aligned} \tag{15}$$

[1] The notation $g = H(f;h)$ means that g is the Hermite interpolant to the function h at the zeros (taken with their multiplicity) of f.

The bounds (14) *and* (15) *can be attained only simultaneously by codes which have all their inner products in the roots of* $f_{2k}^{n,\ell,s}(t)$ *and which are the* $2k$*-designs in* H_q^n.

5. Test functions and refinements by polynomials of higher degrees. In 1998 Boyvalenkov and Danev [6] (see also [7, 5]) introduced the test-functions (16) in the context of maximal codes in polynomial metric spaces (which include H_q^n). Let n and M be fixed and $\tau = \tau(n, M)$ such that $M \in (R_q(n, \tau), R_q(n, \tau+1)]$. As explained in the end of Section 2, the equation $L_\tau(n, s) = M$, $s = \alpha_{k-1}$ or β_k, defines all necessary parameters (α_i, ρ_i) or (β_i, γ_i), respectively.

Let $j \geq \tau + 1$ be a positive integer. The test-functions in n and s are defined as follows.

$$P_j(n,s) := \begin{cases} \dfrac{1}{M} + \displaystyle\sum_{i=0}^{k-1} \rho_i Q_j^{(n,q)}(\alpha_i) & \text{for} \quad s \in \mathcal{I}_{2k-1} \\ \dfrac{1}{M} + \displaystyle\sum_{i=0}^{k} \gamma_i Q_j^{(n,q)}(\beta_i) & \text{for} \quad s \in \mathcal{I}_{2k}. \end{cases} \tag{16}$$

The next theorem shows that the test-functions $P_j(n, s)$ give necessary and sufficient conditions for existence of improving polynomials of higher degrees satisfying the condition $f(t) \leq h(t)$ in $[-1, 1]$.

Theorem 5.1 ([8, Theorem 5.1]). *Let* h *be strictly absolutely monotone function. The bounds* (6) *can be improved by a polynomial of degree at least* $\tau+1$ *from* $\mathcal{A}_{n,M,h}$, *satisfying* $f(t) \leq h(t)$ *in* $[-1, 1]$ *if and only if* $P_j(n, s) < 0$ *for some* $j \geq \tau + 1$. *Furthermore, if* $P_j(n, s) < 0$ *for some* $j \geq \tau + 1$, *then* (6) *can be improved by a polynomial from* $\mathcal{A}_{n,M,h}$, *satisfying* $f(t) \leq h(t)$ *in* $[-1, 1]$ *of degree exactly* j.

Similarly, we can define the corresponding test-functions which give necessary and sufficient conditions for existence of improving polynomials of higher degrees from $\mathcal{A}_{n,\ell,s}$ or $\mathcal{A}_{n,\ell;h}$, satisfying the condition $f(t) \leq h(t)$ in $[\ell, 1]$, and then we can improve the bounds (14) and (15).

We refer to Levenshtein bounds (Theorem 3.5 and Theorem 4.6) and the universal lower bounds on energy (Theorem 4.1 and Theorem 4.7) obtained by Boyvalenkov, Dragnev, Hardin, Saff and the author [8, 9], as *"first level"* bounds. Then we call *"second level"* bounds any improvements of these bounds by polynomials of higher degrees. A framework for obtaining the "second level" bounds in the Euclidean case was developed in [10], but the "second level" in the Hamming case, described in this survey, is still to be claimed.

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УНИВЕРСАЛНИ ГРАНИЦИ ЗА МОЩНОСТТА И ЕНЕРГИЯТА НА КОДОВЕ В ХЕМИНГОВИ ПРОСТРАНСТВА

Мая Стоянова

В статията се представят систематизирано резултати за универсалните граници за размерността и енергията на кодовете в Хемингови пространства. Универсалността означава, че границите са валидни за голям клас от потенциални функции (най-важните от тях са в сила за абсолютно монотонни потенциали). Освен това, като се използват променливи по знак мерки, положително определени до дадена степен, са получени горни граници от тип Левенщейн за мощността на кодовете с дадено минимално и максимално разстояние и универсални долни граници на потенциалната енергия за кодове с дадено максимално разстояние и мощност. Представените методи и техники могат да се разглеждат като обобщение и продължение на разработената от Левенщейн теория.

Ключови думи: Пространства на Хеминг, граници на Левенщейн[2], потенциални функции, енергия на код, кодове, коригиращи грешки, τ-дизайни, променливи по знак мерки.

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[2] По молба на автора оставяме фамилията на руския математик Левенщайн (рус. Владимир Иосифович Левенштейн) в негова памет, както се произнася на руски. (бел. ред.)