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ON SMALL NON-UNIFORM HYPERGRAPHS WITHOUT PROPERTY B*

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For a given hypergraph $H = (V, E)$ consider the sum $q(H)$ of $2^{-|e|}$ over $e \in E$. Consider the class of hypergraphs whose smallest edge is of size n and for which every 2-colouring has a monochromatic edge. Let $q(n)$ be the smallest value of $q(H)$ in this class.

We provide a survey of the known bounds on $q(n)$ and make some minor refinements.

1. Introduction. A hypergraph $H = (V, E)$ is a finite set of vertices V and a set of edges E where each edge is a set of at least two vertices. A 2-colouring of H is an assignment of colour blue or red to each vertex in H. A 2-colouring is *proper* if each edge in H is not monochromatic. We say that H is *2-colourable* if it admits a proper 2-colouring. A hypergraph is said to be *n-uniform* if all of its edges have cardinality n. So a graph is just a 2-uniform hypergraph.

A famous Erdős–Hajnal problem is find the minimum number of edges $m(n)$ in an n-uniform hypergraph that is not 2-colourable. The best known asymptotic bounds are

$$c\sqrt{\frac{n}{\ln n}}2^n \leq m(n) \leq (1 + o(1))\frac{e \ln 2}{4}n^2 2^n,$$

for a positive constant c. The lower bound was proved by Radhakrishnan and Srinivasan [10] and then another proof was given by Cherkashin and Kozik [4]; the upper bound is due to Erdős [7] and stays without improvements from 1963. A survey [12] is devoted to this problem and related topics.

Now let us pass to a non-uniform case. For a given hypergraph $H = (V, E)$ define the quantity

$$q(H) := \sum_{e \in E} 2^{-|e|}.$$

Note that $q(H)$ is the expectation of red edges in a random red-blue colouring of V in which vertices get red colour with probability $1/2$ independently on each other, and thus $q(H)$ is twice smaller than the expectation of monochromatic edges in such a colouring.

Erdős [6] asked in 1963 whether the function $q(n)$ is unbounded, where $q(n)$ is the minimal value of $q(H)$ over non-2-colourable hypergraphs H with the minimal size of

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edge n. Beck [3] in 1978 proved that $q(n) \geq c \log^* t$ for $\log^*$ being the iterated logarithm and some positive constant c.

Then in 2008 Lu [8] announced a proof of a bound $q(n) \geq c\dfrac{\ln n}{\ln \ln n}$ but it turned out to work only for simple hypergraphs (a hypergraph is *simple* if every pair of edges shares at most 1 vertex). Shabanov [15] improved the lower bound for the class of simple hypergraphs to $c\sqrt{n}$ (he also made some refinements for hypergraphs with girth bounded from below in [14]). The best known asymptotic bounds on $q(n)$

$$c \ln n \leq q(n) \leq (1 + o(1))\frac{e \ln 2}{4} n^2,$$

where the lower bound is proved by Duraj, Gutowski, and Kozik [5] and the upper bound is a direct consequence of the Erdős upper bound on $m(n)$ and a straightforward estimate $q(n) \leqslant m(n) \cdot 2^{-n}$.

The main contribution of this note is a twice better asymptotic upper bound on $q(n)$.

Theorem 1.1. *Let n be an integer. Then*

$$q(n) \leq (1 + o(1))\frac{e \ln 2}{8} n^2.$$

The proof is based on (probabilistic) alteration method, the result being a random hypergraph which is the union of an n-uniform hypergraph and an $[n^2/4]$-uniform hypergraph. Akhmejanova [2] refined some lower bounds on $q(n)$ for hypergraphs with only two different edge cardinalities. Radhakrishnan and Srinivasan [11] refined some lower bounds on $q(n)$ for the class of hypergraphs which locally have edges of comparable size.

2. The case of small n. The values of $m(n)$ are known only for $n \leq 4$. We have $m(2) = 3$ with the only example of a triangle graph; and $m(3) = 7$ with the only example of the Fano plane. The best known upper bounds for small n are reached by explicit examples; the current situation is outlined in Aglave, Amarnath, Shannigrahi and Singh [1]. No example of $q(n) < 2^{-n}m(n)$ is known. In this section we present an example of a hypergraph H with the smallest edge with 4 elements such that

$$2^{-4}m(4) < q(H) < 2^{-4}(m(4) + 1),$$

and the structure of H completely differs from the known examples for $m(4)$.

Now focus on $n = 4$. In this case Seymour [13] and Toft [16] independently showed that $m(4) \leqslant 23$. They used the example of a hypergraph on 11 vertices with the following edges:

$$\{1,2,9,10\}, \quad \{3,4,9,10\}, \quad \{5,6,9,10\}, \quad \{7,8,9,10\},$$

$$\{1,2,9,11\}, \quad \{3,4,9,11\}, \quad \{5,6,9,11\}, \quad \{7,8,9,11\},$$

$$\{1,2,10,11\}, \quad \{3,4,10,11\}, \quad \{5,6,10,11\}, \quad \{7,8,10,11\},$$

$$\{1,3,5,8\}, \quad \{1,3,6,7\}, \quad \{1,4,5,7\}, \quad \{1,4,6,7\}, \quad \{1,4,6,8\},$$

$$\{2,3,5,7\}, \quad \{2,3,6,7\}, \quad \{2,3,6,8\}, \quad \{2,4,5,7\}, \quad \{2,4,5,8\}, \quad \{2,4,6,8\}.$$

Östergård [9] shows by a complicated computer search that $m(4) = 23$, and there is only one example on at most 11 vertices.

We provide an example of a hypergraph H with sixteen vertices, twenty 4-edges and

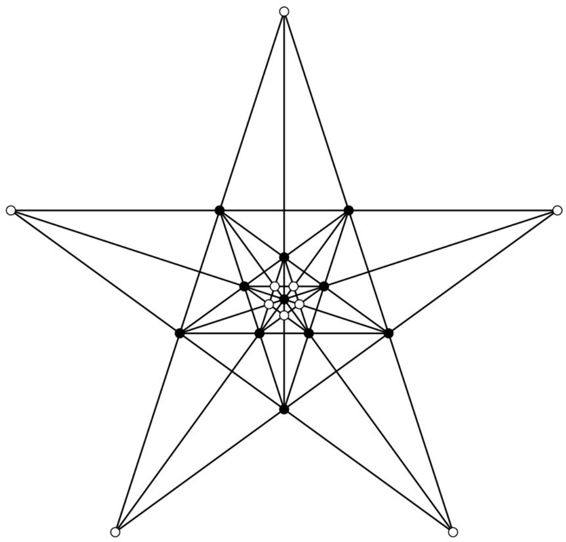

Fig. 1. An explicit definition of H_4. Points are vertices and lines are edges. In this picture, opposite white points are the same. For example, the vertical line appears to contain 5 points, but the 2 white points are the same, so it just contains 4 points

sixty 8-edges which is not 2-colourable. It means that $q(H) = \frac{95}{64} = \frac{23}{16} + \frac{3}{64} < \frac{24}{16}$ which means that this is better than any known 4-graph, except shows the Seymour–Toft graph.

By construction H consists of a 4-uniform part H_4 and an 8-uniform part H_8. The 4-uniform part H_4 is an affine plane over $GF(4)$. It may be also defined in an explicit way, see Fig. 1 (given as an example in a discussion in math.stackexchange.com by the user Matt). A direct computation shows that H_4 has 120 proper 2-colourings and every proper 2-colouring has exactly 8 red and 8 blue vertices[1]. Thus these colourings form 60 "opposite" pairs, i.e. colourings in a pair can be obtained from each other by swapping the colours. Then taking red vertices of one member from each pair as an 8-edge one gets $|E(H_8)| = 60$ and $H_4 \cup H_8$ has no proper 2-colouring as desired.

3. Proof of Theorem 1.1. To avoid rounding in calculations, assume that n is even; the case of odd n is analogues. Consider a set of vertices V with the cardinality $v = n^2/2$, and choose m random edges uniformly and independently; the number m will be specified later. Fix a colouring C; clearly the probability of the event that a randomly chosen edge is monochromatic is equal to

$$p := \frac{\binom{v_1}{n} + \binom{v_2}{n}}{\binom{v}{n}},$$

where v_1 and v_2 denote the numbers of vertices of the first and second colour, respectively. Hence, since the edges are chosen independently, the probability that after choosing m

[1] K. Vorob'ev mentioned that the blue sets of those colourings form a 3-(16,8,12) design.

random independent edges the colouring C is proper is $(1-p)^m$. Let

$$q := \frac{2\binom{v/2}{n}}{\binom{v}{n}}.$$

It is well-known that

$$q = (1+o(1))\frac{2}{e\cdot 2^n}.$$

Note that $p \geq q$ because of the convexity of the sequence $\left\{\binom{t}{n}\right\}_{t\geq 0}$. Since the total number of colourings is $2^{n^2/2}$, and the probability that a fixed colouring is proper is bounded by $(1-q)^m$, the expectation of the number of proper colouring is at most

$$2^{n^2/2}(1-q)^m < e^{\ln 2\cdot n^2/2 - qm}.$$

(We use a standard inequality $1-t < e^{-t}$, $t > 0$.)

To get the Erdős upper bound one should take

$$m = (1+o(1))\frac{e\ln 2}{4}n^2 2^n$$

and check that such a choice leads to $\ln 2\cdot n^2/2 - qm < 0$ which means that with a positive probability a random graph with m edges has no proper 2-colouring.

For our purpose we need a twice smaller number of edges, i.e. $m' = m/2$; then the expectation of the number of proper colouring is at most $2^{n^2/4}$, so with a positive probability a random graph H_1 with m' edges of size n has at most $2^{n^2/4}$ proper colourings. For each such a colouring C we consider an edge e_C of size $n^2/4$ which is monochromatic in C. Let H_2 be a hypergraph on the same vertex set V consisting of all such edges e_C, and let H be the union of H_1 and H_2. Then

$$q(H) = q(H_1) + q(H_2) \leq (1+o(1))\frac{e\ln 2}{8}n^2 + 1 = (1+o(1))\frac{e\ln 2}{8}n^2,$$

as desired.

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МАЛКИ НЕРАВНОМЕРНИ ХИПЕРГРАФИ БЕЗ СВОЙСТВО B

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За даден хиперграф $H = (V, E)$ нека да разгледаме сумата $q(H)$ на $2^{-|e|}$ върху $e \in E$. Разглеждаме класа хиперграфи, чието най-малко ребро е с тегло n и и за които всяко 2-оцветяване има едноцветно ребро. Нека $q(n)$ е най-малката стойност на $q(H)$ в този клас. Представяме преглед на известните граници на $q(n)$ и правим някои малки подобрения.
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