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ON THE DISTRIBUTION OF $\alpha p^2 + \beta$ MODULO ONE FOR PRIMES p SUCH THAT $p + 2$ HAS NO MORE TWO PRIME DIVISORS*

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A classical problem in analytic number theory is to study the distribution of fractional part $\alpha p^k + \beta$, $k \geq 1$ modulo 1, where α is irrational and p runs over the set of primes. We consider the subsequence generated by the primes p such that $p + 2$ is an almost-prime (the existence of infinitely many such p is another topical result in prime number theory) and prove that its distribution has a similar property.
Keywords: linear sieve, almost primes, distribution modulo one.

РАЗПРЕДЕЛЕНИЕ НА ДРОБНИТЕ ЧАСТИ НА $\alpha p^2 + \beta$ ПО МОДУЛ 1 ЗА ПРОСТИ ЧИСЛА p, ЗА КОИТО $p + 2$ ИМА НЕ ПОВЕЧЕ ОТ ДВА ПРОСТИ ДЕЛИТЕЛЯ

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Класически проблем в аналитичната теория на числата е проблемът за разпределението на дробните части на числата αp^k, $k \geq 1$, където α е ирационално и p пробягва множеството на простите числа. Ние разглеждаме подмножество на множеството на простите числа p, за които $p + 2$ е почти просто и доказваме, че тяхното разпределение има свойства подобни на тези на разпределението на простите числа.
Ключови думи: линейно решето, почти прости, разпределение по модул 1.

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1. Introduction and statements of the result. The famous prime twins conjecture states that there exist infinitely many primes p such that $p+2$ is a prime too. This hypothesis is still unproved but in 1973 Chen [2] proved that there are infinitely many primes p for which $p+2 = \mathrm{P}_2$. (As usual P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

Let α be irrational real number and $\|x\|$ denote the distance from x to the nearest integer. The distribution of fractional parts of the sequence αn^k, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ was first considered by Hardy, Littlewood [5] and Weyl [19]. The problem of distribution of the fractional parts of αp^k, where p denotes a prime, first was considered by Vinogradov (see Chapter 11 of [17] for the case $k=1$, [18] for $k \geq 2$), who showed that for any real β there are infinitely many primes p such that

$$\|\alpha p + \beta\| < p^{-\theta}, \tag{1}$$

where $\theta = 1/5 - \varepsilon$, $\varepsilon > 0$ is arbitrary small. After that many authors improved the upper bound of the exponent θ. The best result is given by Matomäki [10] with $\theta = 1/3 - \varepsilon$. Another interesting problem is the study of the distribution of the fractional part of αp^k with $2 \leq k \leq 12$, such Baker and Harman [1], Wong [1] etc. For $2 \leq k \leq 12$ the best result is due to Baker and Harman [1].

In [13] Todorova and Tolev considered the primes p such that $\|\alpha p + \beta\| < p^{-\theta}$ and $p+2 = P_r$ and prove existence of such primes with $\theta = 1/100$ and $r = 4$. Later Matomäki [10] and San Ying Shi [11] have shown that this actually holds whit $p+2 = P_2$ and $\theta = 1/1000$ and $\theta = 1.5/100$ respectively.

In [12] Shi and Wu proved existence of infinitely many primes p such that $\|\alpha p^2 + \beta\| < p^{-\theta}$ and $p+2 = P_4$ with $0 < \theta < 2/375$. In 2021 Xue, Li and Zhang [14] improved the result of Shi and Wu with $0 < \theta < 10/1561$.

In this paper we evaluate exponential sums over well-separated numbers and improve the results of Shi, Wu and Xue, Li and Zhang.

We will say that d is a well-separable number of level $D \geq 1$ if for any $H, S \geq 1$ with $HS = D$, there are integers $h \leq H$, $s \leq S$ such that $d = hs$.

Theorem 1. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies conditions*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad a \in \mathbb{Z}, \ q \in \mathbb{N}, \quad (a,q) = 1, \quad q \geq 1, \tag{2}$$

K and D are defined by (8), $\lambda(d)$ are complex numbers defined for $d \leq D$,

$$\lambda(d) \ll \tau(d) \quad \text{and} \quad \lambda(d) \neq 0 \quad \text{if} \quad d \quad \text{is well-separable number of level} \quad D, \tag{3}$$

$c(k) \ll 1$ are complex numbers, $0 < |k| \leq K$. Then for any arbitrary small $\varepsilon > 0$ and $b \in \mathbb{Z}$ for the sum

$$W(x) = \sum_{d \leq D} \lambda(d) \sum_{1 \leq |k| \leq K} c(k) \sum_{\substack{n \sim x \\ n \equiv b\,(d)}} e\big((\alpha n^2 + \beta)k\big)\Lambda(n) \tag{4}$$

we have

$$W \ll x^{\varepsilon}\left(\frac{xK}{\Delta^{\frac{1}{32}}} + \frac{xK}{q^{\frac{1}{32}}} + \frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{4}}} + x^{\frac{71}{72}}\Delta^{\frac{33}{64}}K + x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}} + x^{\frac{1}{2}}\Delta^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{5}$$

Remark 1. It is obvious that the Theorem 1 is true if function $\lambda(d)$ is well-factorable.

Lemma 1. *Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies conditions (2), sum $W(x)$ is defined by (4), $\lambda(d)$ are complex numbers defined for $d \leq D$ and satisfying (3) and (8), $c(k) \ll 1$ are* *complex numbers* $0 < |k| \le K$. *Then there exist a sequence*

$$\{x_j\}_{j=1}^{\infty}, \ \lim_{j\to\infty} x_j = \infty,$$

such that

$$W(x_j) \ll x_j^{1-\omega}, \quad j = 1, 2, 3, \ldots$$

for any $\omega > 0$.

Theorem 2. *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *satisfies conditions* (2), $\beta \in \mathbb{R}$ *and let*

$$0 < \ \theta < \ \frac{1}{1296} - \eta,$$

where η *is arbitrary small fixed number. Then there are infinitely many primes p satisfying* $p + 2 = \mathrm{P}_2$ *and such that*

$$\|\alpha p^2 + \beta\| < p^{-\theta}. \tag{6}$$

2. Notation. Let x be a sufficiently large real number and θ and ρ be real constants satisfying

$$0 < \theta < \frac{1}{1296}, \qquad \rho > 32\theta, \qquad \rho + \frac{64\theta}{33} < \frac{8}{297} \qquad \rho + 10\theta > \varepsilon. \tag{7}$$

We shall specify ρ and θ latter. We put

$$\begin{gathered} \delta = \delta(x) = x^{-\theta}, \quad K = \delta^{-1}\log^2 x, \\ \Delta = x^{\rho}, \quad D = \frac{x^{1/2}}{\Delta K^4}. \end{gathered} \tag{8}$$

By p and q we always denote primes. As usual $\varphi(n)$, $\mu(n)$, $\Lambda(n)$ denote respectively Euler's function, Möbius' function and Mangoldt's function. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \ldots m_k = n$ in natural numbers $m_1, \ldots, m_k$ and $\tau_2(n) = \tau(n)$. Let $(m_1, \ldots, m_k)$ and $[m_1, \ldots, m_k]$ be the greatest common divisor and the least common multiple of $m_1, \ldots, m_k$ respectively. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. As usual, $\|y\|$ denotes the distance from y to the nearest integer, $e(y) = e^{2\pi i y}$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$ and $k \sim K$ means $K/2 \le k < K$. The letter ε denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^{\varepsilon}\log x \ll x^{\varepsilon}$.

3. Some lemmas.

Lemma 2. *Let* k, l, m, $n \in \mathbb{N}$; $X, \varepsilon \in \mathbb{R}$; $X \ge 2$, $k \ge 2$ *and* $\varepsilon > 0$. *Then*

$$\text{(i)} \qquad \sum_{n \le X} \left(\tau_k(n)\right)^l \ll_{k,l} X(\log X)^{k^l - 1} \ ;$$

$$\text{(ii)} \qquad \tau_k(n) \ll_{k,\varepsilon} n^{\varepsilon} \ .$$

Proof. See [16], ch. 3. □

Lemma 3. *Let* $X \ge 1$ *and* α *satisfied conditions* (2) *and* a, $d \in \mathbb{N}$. *Then*

$$\Bigg| \sum_{\substack{n \le X \\ n \equiv a \pmod d}} e(\alpha n) \Bigg| \ll \min\left(\frac{X}{d}, \frac{1}{\|\alpha d\|}\right)$$

Proof. See [9], ch.6, §2. □

Lemma 4. *Let* X, $Y \in \mathbb{R}$; $k \in \mathbf{N}$; X, $Y \ge 1$; $k \ \ge \ 2$ *and* α *satisfied conditions* (2).

Then

$$\sum_{n\le X} \min\Big(\frac{XY}{n}, \frac{1}{\|\alpha n\|}\Big) \ll XY\Big(\frac{1}{q}+\frac{1}{Y}+\frac{q}{XY}\Big)\log(2Xq).$$

Proof. See Lemma 2.2 from [15], ch. 2,§2.1. $\square$

Lemma 5. *Let $\mu, \zeta \in \mathbb{N}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and α satisfy conditions* (2)*. Then for every arbitrary small $\varepsilon > 0$ the inequality*

$$\sum_{m\sim M} \tau_\mu(m) \sum_{j\sim J} \tau_\zeta(j) \min\left\{\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right\}$$
$$\ll x^\varepsilon\left(MJ + \frac{x}{M^{3/2}} + \frac{x}{Mq^{1/2}} + x^{1/2}q^{1/2}\right)$$

is fulfilled.

Proof. See Lemma 8, [10]. $\square$

Lemma 6. *If $d|P(z)$, $z < D^{1/2}$, $\lambda^\pm$ are Rosser's weights and either $\lambda^+(d)\ne 0$ or $\lambda^-(d)\ne 0$ then d is well-separated number.*

Proof. See Lemma 12.16, [3] $\square$

Theorem 3. *Let $2 \le z \le D^{1/2}$ and $s = \dfrac{\log D}{\log z}$. If*

$$\begin{cases} \mathcal{A}_d = \dfrac{\omega(d)}{d}x + r(x, d) & \text{if} \quad \mu(d)\ne 0 \\ \displaystyle\sum_{z_1\le p<z_2} \frac{\omega(p)}{p} = \log\left(\frac{\log z_2}{\log z_1}\right) + O\Big(\frac{1}{\log z_1}\Big), & z_2 > z_1 \ge 2 \end{cases}$$

where $\omega(d)$ is a multiplicative function, $0 < \omega(p) < p$, $x > 1$ is independent of d. Then

$$xV(z)\bigg(f(s) + O\Big(\frac{1}{(\log D)^{1/3}}\Big)\bigg) \le S(\mathcal{A}, z) \le xV(z)\bigg(F(s) + O\Big(\frac{1}{(\log D)^{1/3}}\Big)\bigg),$$

where d are well-separated numbers of level D, $f(s)$, $F(s)$ are determined by the following differential-difference equations

$$F(s) = \frac{2e^\gamma}{s},\ f(s) = 0 \quad \text{if} \quad 0 < s \le 2$$
$$(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1) \quad \text{if} \quad s > 2,$$

where γ denote the Euler's constant.

4. Auxiliary results.

Lemma 7. *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ satisfied conditions* (2)*, $M, S, J, x \in \mathbb{R}^+$, $x > M^3S^2J$ and $\mu, \sigma, \zeta \in [2, \infty)\cap\mathbb{N}$,*

$$G = \sum_{m\sim M} \tau_\mu(m) \sum_{s\sim S} \tau_\sigma(s) \sum_{j\sim J} \tau_\zeta(j) \min\left\{\frac{x}{m^3s^2j}, \frac{1}{\|\alpha m^3 s^2 j\|}\right\}.$$

Then for any $\varepsilon > 0$ the inequalities

$$G \ll x^\varepsilon\left(MSJ + \frac{x}{M^{\frac{9}{4}}S} + \frac{x}{M^2S^{\frac{9}{8}}} + \frac{x}{M^2Sq^{\frac{1}{8}}} + \frac{x^{\frac{7}{8}}q^{\frac{1}{8}}}{M^2S}\right) \tag{9}$$

and

$$G \ll x^{\varepsilon}\left(MSJ + \frac{x}{M^{\frac{9}{4}}S^{\frac{3}{4}}} + \frac{x}{M^2 S^{\frac{3}{4}} q^{\frac{1}{4}}} + \frac{x^{\frac{3}{4}} q^{\frac{1}{4}}}{M^2 S^{\frac{3}{4}}}\right) \tag{10}$$

are fulfilled.

Proof. Our proof is similar to proof of Lemma 8, [10]. Let

$$H = \frac{x}{M^3 S^2 J}. \tag{11}$$

If $H \le 2$, then trivially from Lemma 2 (iv) we get

$$G \ll x^{\varepsilon} MSJ. \tag{12}$$

So we can assume that $H > 2$. From Lemma 2 (iv) it is obviously that

$$G \ll x^{\varepsilon} \sum_{m \sim M} \sum_{s \sim S} \sum_{j \sim J} \min\left\{\frac{x}{m^3 s^2 j}, \frac{1}{\|\alpha m^3 s^2 j\|}\right\}.$$

We apply the Fourier expansion to the function $\min\left\{\frac{x}{m^3 s^2 j}, \frac{1}{\|\alpha m^3 s^2 j\|}\right\}$ and get

$$\min\left\{\frac{x}{m^3 s^2 j}, \frac{1}{\|\alpha m^3 s^2 j\|}\right\} = \sum_{0<|h|\le H^2} w(h) e(\alpha m^3 s^2 j h) + O(\log x),$$

where

$$w(h) \ll \min\left\{\log H, \frac{H}{|h|}\right\}. \tag{13}$$

Then

$$G| \ll x^{\varepsilon} \sum_{0<|h|\le H^2} |w(h)| \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m^3 s^2 j h) \right| + MSJ \log x. \tag{14}$$

So if

$$G(H_0) = \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m^3 s^2 j h) \right|.$$

then using (13) we have

$$G \ll x^{\varepsilon}\left(MSJ + \max_{1 \le H_0 \le H_1} G(H_0) + \max_{H_1 \le H_0 \le H^2} \frac{H}{H_0} G(H_0)\right). \tag{15}$$

We shall evaluate the sum $G(H_0)$. Applying the Cauchy–Schwarz inequality we obtain

$$G^2(H_0) \ll x^{\varepsilon} H_0 JS \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m^3 s^2 j h) \right|^2$$

$$\ll x^{\varepsilon} H_0 JS \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \sum_{m_1 \sim M} \sum_{m_2 \sim M} e\big(\alpha (m_1^3 - m_2^3) s^2 j h\big).$$

Substituting $m_1 = m_2 + t$, where $0 \le |t| \le M$ we get

$$G^2(H_0) \ll x^{\varepsilon}\left(H_0^2 J^2 S^2 M + H_0 JS G_1(H_0)\right), \tag{16}$$

where

$$G_1(H_0) = \sum_{h\sim H_0} \sum_{s\sim S} \sum_{j\sim J} \sum_{0<|t|<M} \Bigg| \sum_{m_2\sim M} e\big(\alpha(3m_2^2 t + 3m_2 t^2)s^2 jh\big)\Bigg|.$$

Applying again the Cauchy–Schwarz inequality we obtain

$$G_1^2(H_0) \ll H_0 JSM \sum_{h\sim H_0} \sum_{s\sim S} \sum_{j\sim J} \sum_{0<|t|<M} \sum_{m_2\sim M} \times \sum_{m_3\sim M} e\big(\alpha(3(m_2^2 - m_3^2)t + 3(m_2 - m_3)t^2)s^2 jh\big).$$

Substituting $m_2 = m_3 + \ell$, where $0 \le |\ell| \le M$ we get

$$G_1^2(H_0) \ll H_0^2 J^2 S^2 M^3 + H_0 JSM \sum_{h\sim H_0} \sum_{s\sim S} \sum_{j\sim J} \sum_{0<|t|<M} \sum_{0<|\ell|<M} \Bigg| \sum_{m_3\sim M} e\big(6\alpha m_3 t\ell s^2 jh\big)\Bigg|.$$

Let $u = 6t\ell hj$. Then using Lemma 3 and Lemma 5 we get

$$G_1^2(H_0) \ll H_0^2 J^2 S^2 M^3 + H_0 JSM \sum_{u\le 24H_0JM^2} \tau_5(u) \sum_{s\sim S} \Bigg| \sum_{m_3\sim M} e\big(\alpha m_3 s^2 u\big)\Bigg|$$

$$\ll H_0^2 J^2 S^2 M^3 + H_0 JSM \sum_{u\ll H_0JM^2} \tau_5(u) \sum_{s\sim S} \min\left\{ \frac{H_0 JS^2 M^3}{s^2 u}, \frac{1}{||\alpha s^2 u||} \right\}. \tag{17}$$

We will estimate the above sum in two ways. Using Lemma 5 we obtain

$$G_1^2(H_0) \ll x^\varepsilon \left(H_0^2 J^2 S^2 M^3 + H_0^2 J^2 S^{\frac{3}{2}} M^4 + \frac{H_0^2 J^2 S^2 M^4}{q^{\frac{1}{2}}} + H_0^{\frac{3}{2}} J^{\frac{3}{2}} SM^{\frac{5}{2}} q^{\frac{1}{2}} \right).$$

So from (16)

$$G(H_0) \quad \ll \quad x^\varepsilon \left(H_0 JSM^{\frac{3}{4}} \ + \ H_0 JS^{\frac{7}{8}} M \ + \ \frac{H_0 JSM}{q^{\frac{1}{8}}} \ + \ H_0^{\frac{7}{8}} J^{\frac{7}{8}} S^{\frac{3}{4}} M^{\frac{5}{8}} q^{\frac{1}{8}} \right). \tag{18}$$

Choosing $H_0 = H$ from (11), (15) and (18) we get (9).

On the other hand we can write the inequality (17) as

$$G_1^2(H_0) \ll H_0^2 J^2 S^2 M^3 + H_0 JSM \sum_{k\ll H_0JS^2M^2} \min\left\{ \frac{H_0 JS^2 M^3}{k}, \frac{1}{||\alpha k||} \right\}$$

and using Lemma 4 and (16) we get

$$G^2(H_0) \ll x^\varepsilon \left(H_0^2 J^2 S^3 M^3 + \frac{H_0^2 J^2 S^3 M^4}{q} + H_0 JSMq \right) \tag{19}$$

Now we choose $H_0 = H$. Then from (11), (15) and (19) the inequality (10) is received. □

5. Proof of Theorem 1. To prove Theorem 1 we shall evaluate the sum W in two ways:

when $x^{8/27}\Delta \le D \le \dfrac{x^{1/2}}{\Delta K^4}$, we will use the Vaughan's identity;

when $D \le x^{8/27}\Delta$, we will use the Heat-Brown identity.

5.1. Evaluation by Vaughan's identity. Let $x^{8/27}\Delta \le D \le \dfrac{x^{1/2}}{\Delta K^4}$ and $0 < |K| \le \delta^{-1}\log^2 x$. First we decompose the sum $W(x)$ into $O(\log^2 x)$ sums of type

$$W = W(x,\, D,\, K) = \sum_{d\sim D} \lambda(d) \sum_{1\le|k|\sim K} c(k) \sum_{\substack{n\sim x\\ n+2\equiv 0\,(d)}} e\big((\alpha n^2+\beta)k\big)\Lambda(n),$$

where $\lambda(d)$ is Roser weight and in particular a necessary condition for $\lambda(d)\neq 0$ is numbers d are squarefree. So from this point on we will use that numbers d are squarefree. Then by Vaughan's identity we can decompose the sum W into $O(\log x)$ type I sums

$$W_1 = \sum_{\substack{d\sim D\\ (a,\,d)=1}} \lambda(d) \sum_{k\sim K} c(k)e(\beta k) \sum_{\substack{m\sim M\\ \ell\sim L\\ mn\equiv a(\mathrm{mod}\ d)}} a(m)e(\alpha(m\ell)^2k)$$

or

$$W_1' = \sum_{\substack{d\sim D\\ (a,\,d)=1}} \lambda(d) \sum_{k\sim K} c(k)e(\beta k) \sum_{\substack{m\sim M\\ \ell\sim L\\ m\ell\equiv a(\mathrm{mod}\ d)}} \log(n)e(\alpha(m\ell)^2k)$$

with $M \le x^{1/3}$ and $O(\log x)$ type II sums

$$W_2 = \sum_{\substack{d\sim D\\ (a,\,d)=1}} \lambda(d) \sum_{k\sim K} c(k)e(\beta k) \sum_{\substack{m\sim M\\ \ell\sim L\\ m\ell\equiv a(\mathrm{mod}\ d)}} a(m)b(\ell)e(\alpha(m\ell)^2k)$$

with $M \in [x^{1/3},\, x^{2/3}]$ and

$$ML \sim x, \quad a(m) \ll \tau_3(m)\log m, \quad b(\ell) \ll \tau_3(\ell)\log \ell \tag{20}$$

5.1.1. Evaluation of type II sums. The proof follows proof of Theorem 1, [10]. As $x^{1/3} \le M,\, L \le x^{2/3}$ and $ML \sim x$ we will consider only the case $x^{1/2} \le M \le x^{2/3}$. The evaluation in the case $x^{1/2} \le L \le x^{2/3}$ is the same. Using that d is well-separated numbers we write $d = hs$, where $(h,s)=1$ as d is squarefree. So the sum W_2 is presented as $O(\log^2 x)$ sums of the type

$$W_2 = \sum_{\substack{h\sim H\\ (h,\,a)=1}} \sum_{\substack{s\sim S\\ (s,\,ah)=1}} \lambda(hs) \sum_{k\sim K} c(k)e(\beta k) \sum_{\ell\sim L} \sum_{\substack{m\sim M\\ m\ell\equiv a(\mathrm{mod}\ hs)}} a(m)b(\ell)e(\alpha(m\ell)^2k).$$

Here

$$h\sim H, \quad s\sim S, \quad D\sim HS \tag{21}$$

and H we will choose later. Applying the Cauchy–Schwarz inequality to W_2 and using and Lemma 2(i) we obtain that

$$W_2^2 \ll x^\varepsilon KHM \sum_{k\sim K} \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{s_1'\sim S\\(s_1',ah)=1}} \lambda(hs_1') \sum_{\substack{s_2'\sim S\\(s_2',ah)=1}} \lambda(hs_2') \times \sum_{\ell_1\sim L} b(\ell_1) \sum_{\ell_2\sim L} b(\ell_2) \sum_{\substack{m\sim M\\ m\ell_1\equiv a(hs_1')\\ m\ell_2\equiv a(hs_2')}} e(\alpha m^2(\ell_1^2-\ell_2^2)k). \tag{22}$$

Let $(s_2',\, s_1') = r$, $s_1' = rs_1$, $s_2' = rs_2$, $r \sim R$, $R \le S$ and $s_1',\, s_2' \sim S/R$. Then

$$W_2^2 \ll x^\varepsilon KHM \sum_{k\sim K} \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{r\sim R\\(r,ah)=1}} \sum_{\substack{s_1\sim S/R\\(s_1,ah)=1}} \lambda(hrs_1) \sum_{\substack{s_2\sim S/R\\(s_2,as_1h)=1}} \lambda(hrs_2) \times \sum_{\ell_1\sim L} b(\ell_1) \sum_{\ell_2\sim L} b(\ell_2) \sum_{\substack{m\sim M\\ m\ell_1\equiv a(hrs_1)\\ m\ell_2\equiv a(hrs_2)}} e(\alpha m^2(\ell_1^2-\ell_2^2)k) = W_{21} + W_{22}, \tag{23}$$

where W_{21} is this one part of above sum for which

$$\begin{aligned} &\ell_1 = \ell_2, \quad s_1 \ne s_2 \quad \text{or}\\ &\ell_1 = \ell_2, \quad s_1 = s_2 = 1 \quad r\sim S \quad \text{or}\\ &\ell_1 \ne \ell_2, \quad s_1 = s_2 = 1 \quad r\sim S \quad \text{or}\\ &\ell_1 \ne \ell_2, \quad s_1 \ne s_2, \quad M < \frac{4HS^2}{R}, \end{aligned}$$

W_{22} is the rest part of sum for W_2^2. To evaluate the sum W_{21} we consider the cases $x^{1/2} \le M \le \dfrac{x}{D}$ and $x^{1/3} \le L \le D$. Then using Lemma 2 we get

$$W_{21} \ll x^\varepsilon\left(xMHK^2 + xD^2K^2 + \frac{x^2HK^2}{D} + \frac{xLD^2K^2}{H}\right). \tag{24}$$

It is clear that for sum W_{22} we have $\ell_1 \ne \ell_2$, $s_1 \ne s_2$, $M > \dfrac{4HS^2}{R}$. From

$$m\ell_1 \equiv a(hrs_1),\ m\ell_2 \equiv a(hrs_2) \quad \text{follows that} \quad \ell_1 \equiv \ell_2(hr).$$

We apply again the Cauchy–Schwarz inequality and get

$$W_{22}^2 \ll \frac{x^{2+\varepsilon}D^2K^3}{R^2} \sum_{k\sim K} \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{r\sim R\\(r,ah)=1}} \sum_{\substack{s_1\sim S/R\\(s_1,ah)=1}} \sum_{\substack{s_2\sim S/R\\(s_2,as_1h)=1}} \times \sum_{\ell_1\sim L} \sum_{\substack{\ell_2\sim L\\ \ell_2\equiv\ell_1(hr)}} \sum_{\substack{m_1\sim M\\ m_1\ell_1\equiv a(hrs_1)\\ m_1\ell_2\equiv a(hrs_2)}} \sum_{\substack{m_2\sim M\\ m_2\ell_1\equiv a(hrs_1)\\ m_2\ell_2\equiv a(hrs_2)}} e(\alpha(m_1^2-m_2^2)(\ell_1^2-\ell_2^2)k).$$

Let W_{221} be this one part of above sum for which $m_1 = m_2$ and W_{222} be this part for which $m_1 \neq m_2$. It is not difficult to see that

$$W_{221} \ll \frac{x^{3+\varepsilon} L D^2 K^4}{H}. \tag{25}$$

Let consider the sum W_{222}. As

$$m_i \ell_1 \equiv a \pmod{hrs_1} \quad \text{and} \quad m_i \ell_2 \equiv a \pmod{hrs_2}, \quad i = 1, 2$$

we get

$$m_1 \equiv m_2 (\bmod\ hrs_1 s_2) \equiv f (\bmod\ hrs_1 s_2), \quad \text{where} \quad f = f(h,\, r,\, s_1,\, s_2,\, \ell_1,\, \ell_2)$$

and $\ell_1 \equiv \ell_2 (\bmod\ hr)$. Let

$$m_1 = m_2 + hrs_1 s_2 t,\ 0 < |t| \le \frac{8MR}{HS^2} \quad \text{and} \quad \ell_1 = \ell_2 + hru,\ 0 < |u| \le \frac{2L}{HR}.$$

Then

$$m_1^2 - m_2^2 = 2m_2 hrs_1 s_2 t + h^2 r^2 s_1^2 s_2^2 t^2 \quad \text{and} \quad \ell_1^2 - \ell_2^2 = hru(2\ell_2 + hru).$$

So using above equalities and Lemma 3 we obtain

$$W_{222} \ll \frac{x^{2+\varepsilon} D^2 K^3}{R^2} \sum_{k \sim K} \sum_{\substack{h \sim H \\ (h,\, a)=1}} \sum_{\substack{r \sim R \\ (r,\, ah)=1}} \sum_{\substack{s_1 \sim S/R \\ (s_1,\, ah)=1}} \sum_{\substack{s_2 \sim S/R \\ (s_2,\, as_1 h)=1}}$$

$$\times \sum_{\ell \sim L} \sum_{0<|u| \le \frac{2L}{HR}} \sum_{0<|t| \le \frac{8MR}{HS^2}} \min\left\{ \frac{M}{hrs_1 s_2},\, \frac{1}{\|2\alpha h^3 r^3 s_1^2 s_2^2 tu\ell k\|} \right\},$$

where $\ell = 2\ell_2 + hru$. We put

$$m = hr, \quad s = s_1 s_2, \quad j = 2tunk,\ j \ll \frac{xLK}{D^2}$$

and it is clear that the sum W_{222} can be represented as a finite number of sums of the type

$$W_{223} = \frac{x^{2+\varepsilon} D^2 K^3}{R^2} \sum_{m \sim HR} \tau(m) \sum_{s \sim \frac{S^2}{R^2}} \tau(s) \sum_{j \ll \frac{xLK}{D^2}} \tau_5(j) \min\left\{ \frac{x^2 K}{m^3 s^2 j},\, \frac{1}{\|\alpha m^3 s^2 j\|} \right\}.$$

Using Lemma 7, (21), (23), (24) and (25) we get

$$W_{223} \ll x^{\varepsilon} \Bigg(x^{\frac12} M^{\frac12} H^{\frac12} K + x^{\frac12} DK + \frac{x^{\frac12} L^{\frac12} DK}{H^{\frac12}} + \frac{x^{\frac34} L^{\frac14} D^{\frac12} K}{H^{\frac14}} + \frac{xH^{\frac12} K}{D^{\frac12}}$$
$$+ \frac{xK}{H^{\frac1{16}}} + \frac{xH^{\frac1{16}} K}{D^{\frac1{16}}} + \frac{xK}{q^{\frac1{32}}} + x^{\frac{15}{16}} K^{\frac{31}{32}} q^{\frac1{32}} \Bigg). \tag{26}$$

According to D, M and L we have

$$W_{223} \ll x^{\varepsilon} \Big(V_1 + V_2 + V_3 + V_4 \Big), \tag{27}$$

where V_1 is the sum with

$$x^{1/2} \le M \le \frac{x}{D}, \quad x^{2/5} \le D \le \frac{x^{1/2}}{\Delta K^4}$$
$$D \le L \le x^{1/2}, \tag{28}$$

V_2 is the sum with

$$\tag{29} \begin{aligned} &\frac{x}{D} \le M < x^{1/3}D^{2/3}, \quad x^{2/5} \le D \le \frac{x^{1/2}}{\Delta K^4} \\ &\frac{x^{2/3}}{D^{2/3}} < L \le D, \end{aligned}$$

V_3 is the sum with

$$\tag{30} \begin{aligned} &x^{1/3}D^{2/3} \le M \le x^{2/3}, \quad x^{2/5} \le D \le \frac{x^{1/2}}{\Delta K^4} \\ &x^{1/3} \le L \le \frac{x^{2/3}}{D^{2/3}} \end{aligned}$$

and V_4 is the sum with

$$\tag{31} \begin{aligned} &x^{1/2} \le M \le x^{2/3}, \quad x^{8/27}\Delta \le D \le x^{2/5} \\ &x^{1/3} \le L \le x^{1/2}. \end{aligned}$$

For sums V_1, V_2, V_3 and V_4 we choose consequently

$$H = \frac{D}{\Delta^{1/2}}, \quad H = \frac{LD^{2/3}}{x^{1/3}}, \quad H = \frac{x^{1/3}}{\Delta}, \quad H = \frac{L^{4/5}D^{9/5}}{x^{4/5}}.$$

and from (26), (28), (29), (30) and (31) we get

$$W_2 \ll \begin{cases} x^{\varepsilon}\left(\dfrac{xK}{\Delta^{\frac{1}{32}}} + + x^{\frac{5}{6}}D^{\frac{1}{3}}\Delta^{\frac{1}{4}}K + \dfrac{xK}{q^{\frac{1}{32}}} + x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}}\right), & \text{if} \quad x^{2/5} \le D \le \dfrac{x^{1/2}}{\Delta K^4}, \\ x^{\varepsilon}\left(\dfrac{x^{\frac{31}{30}}K}{D^{\frac{9}{80}}} + \dfrac{xK}{q^{\frac{1}{32}}} + x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}}\right), & \text{if} \quad x^{8/27}\Delta \le D \le x^{2/5}. \end{cases}$$

So

$$\tag{32} W_2 \ll x^{\varepsilon}\left(\frac{xK}{\Delta^{\frac{1}{32}}} + x^{\frac{5}{6}}D^{\frac{1}{3}}\Delta^{\frac{1}{4}}K + \frac{xK}{q^{\frac{1}{32}}} + x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}}\right).$$

5.1.2. Evaluation of type I sums. In this case we have that $L > x^{\frac{2}{3}}$ and $M < x^{\frac{1}{3}}$. Again we will use that d is well-separated numbers. So we can write $d = hs$ with h and s satisfying conditions (21) and we will choose H later. So the sum W_1 is presented as $O(\log^2 x)$ sums of the type

$$W_1 = \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{s\sim S\\(s,a)=1}} \lambda(hs) \sum_{k\sim K} c(k)e(\beta k) \sum_{\ell\sim L} \sum_{\substack{m\ell\sim M\\ m\ell\equiv a (\text{mod } d)}} a(m)e(\alpha(m\ell)^2k).$$

Working in the same way as in the evaluation of the sum W_2 see (22), we get

$$\tag{33} \begin{aligned} W_1^2 \ll x^{\varepsilon}KHM \sum_{k\sim K} \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{r\sim R\\(r,ah)=1}} \sum_{\substack{s_1\sim S/R\\(s_1,ah)=1}} \lambda(hrs_1) \sum_{\substack{s_2\sim S/R\\(s_2,as_1h)=1}} \lambda(hrs_2) \\ \times \sum_{\ell_1\sim L} b(\ell_1) \sum_{\ell_2\sim L} b(\ell_2) \sum_{\substack{m\sim M\\ m\ell_1\equiv a(hrs_1)\\ m\ell_2\equiv a(hrs_2)}} e(\alpha m^2(\ell_1^2-\ell_2^2)k) \\ = W_{11} + W_{12} + W_{13}, \end{aligned}$$

where W_{12} is this one part of above sum for which

$$\ell_1 = \ell_2,$$

W_{13} is this one part of above sum for which

$$\ell_1 \neq \ell_2, \quad s_1 = s_2 = 1 \quad r \sim S$$

and W_{11} is the rest part of sum for W_1^2. Using that $L > x^{\frac{2}{3}}$ and $M < x^{\frac{1}{3}}$ we get

$$W_{12} \ll x^{\varepsilon}.xMHK^2. \tag{34}$$

For the sum W_{13} we get

$$W_{13} \ll x^{\varepsilon} KHM \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,a)=1}} \sum_{m\sim M} \left| \sum_{\substack{\ell_i \sim L\\ m\ell_i \equiv a(d)\\ i=1,1}} e(\alpha m^2(\ell_1^2 - \ell_2^2)k \right|.$$

As $\ell_1 \equiv \ell_2 (\bmod\ d)$ we put

$$\ell_1 = \ell_2 + du,\ 0 < |u| \ll \frac{L}{D}.$$

So

$$W_{13} \ll x^{\varepsilon} KHM \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,a)=1}} \sum_{m\sim M} \sum_{u \ll \frac{L}{D}} \left| \sum_{\substack{\ell_2 \sim L\\ m\ell_2 \equiv a(d)}} e(2\alpha m^2 \ell_2 udk) \right|$$

and from Lemma 3 we get

$$W_{13} \ll x^{\varepsilon} KHM \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,a)=1}} \sum_{m\sim M} \sum_{u \ll \frac{L}{D}} \min\left\{ \frac{x^2K}{m^2d^2(2uk)}, \frac{1}{\|\alpha m^2 d^2 (2uk)\|}. \right\}$$

The above sum can be represented as a finite number of sums of the type

$$W_{14} \ll x^{\varepsilon} KHM \sum_{z \sim MD} \tau(z) \sum_{t \ll \frac{LK}{D}} \tau_3(t) \min\left\{ \frac{x^2K}{z^2t}, \frac{1}{\|\alpha z^2 t\|}. \right\}$$

Using Lemma 5 and $ML \sim x$ we obtain

$$W_{13}^{\frac{1}{2}} \ll x^{\varepsilon}\left(x^{\frac{1}{2}} M^{\frac{1}{2}} H^{\frac{1}{2}} K + \frac{x^{\frac{3}{4}} L^{\frac{1}{4}} H^{\frac{1}{2}} K}{D^{\frac{3}{4}}} + \frac{xH^{\frac{1}{2}}K}{D^{\frac{1}{2}} q^{\frac{1}{4}}} + \frac{x^{\frac{1}{2}} H^{\frac{1}{2}} q^{\frac{1}{4}} K^{\frac{3}{4}}}{D^{\frac{1}{2}}} \right) \tag{35}$$

Using analogous reasoning for the sum W_{11} we get

$$W_{11} \ll x^{\varepsilon} KHM \sum_{k\sim K} \sum_{\substack{h\sim H\\(h,a)=1}} \sum_{\substack{r\sim R\\(r,ah)=1}} \sum_{\substack{s_i \sim S/R\\(s_1s_2,ah)=1}} \times \sum_{m\sim M} \sum_{u \ll \frac{L}{HR}} \left| \sum_{\substack{\ell_2 \sim L\\ m\ell_2 \equiv a(hrs_2)\\ m(\ell_2+uhr)\equiv a(s_1)}} e(2\alpha m^2 \ell_2 uhrk) \right| \tag{36}$$

and from Lemma 3 we obtain

$$W_{11} \ll x^{\varepsilon} KHM \sum_{k\sim K} \sum_{h\sim H} \sum_{r\sim R} \sum_{\substack{s_i\sim S/R\\ i=1,2}} \sum_{m\sim M} \sum_{u\ll \frac{L}{HR}} \min\left\{ \frac{L}{hrs_1s_2}, \frac{1}{\|\alpha m^2h^2r^2.2s_1s_2uk\|} \right\}.$$

Applying Lemma 5 and using that $ML\sim x$ we get

$$W_{11}^{\frac{1}{2}} \ll x^{\varepsilon}\left(\frac{x^{\frac12}DK}{H^{\frac12}} + \frac{x^{\frac34}L^{\frac14}K}{H^{\frac14}} + \frac{xK}{q^{\frac14}} + x^{\frac12}q^{\frac14}K^{\frac34}\right) \tag{37}$$

Choosing $H=\dfrac{D}{M^{\frac12}}$ from (34), (35), (37), (22) and (5) follows

$$W_1 \ll x^{\varepsilon}\left(\frac{xK}{D^{\frac14}} + \frac{xK}{q^{\frac14}} + x^{\frac12}K^{\frac34}q^{\frac14}\right). \tag{38}$$

From (32) and (38) it follows that in the case $x^{8/27}\Delta \le D \le \dfrac{x^{1/2}}{\Delta K^4}$ the estimate

$$W \ll x^{\varepsilon}\left(\frac{xK}{\Delta^{\frac{1}{32}}} + +x^{\frac56}D^{\frac13}\Delta^{\frac14}K + \frac{xK}{q^{\frac{1}{32}}} + x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}}\right) \tag{39}$$

is fulfilled.

5.2. Evaluation by Heat-Brown's identity. Let $D\le x^{8/27}\Delta$. We decompose the sum $W(x)$ into $O(\log^2 x)$ as in (4). Using Heath-Brown's identity [6] with parameters

$$P=x/2,\ P_1=x,\ u=\frac{x^{\frac13}}{2^{21}D^{\frac{1}{32}}},\ v=2^7x^{\frac13},\ w=2^7x^{\frac13}D^{\frac{1}{64}}. \tag{40}$$

we decompose the sum W as a linear combination of $O(\log^6 N)$ sums of first and second type. The sums of the first type are

$$W_1 = \sum_{d\le D}\lambda(d) \sum_{0<|k|\le K} c(k)e(\beta k) \sum_{M<m\le M_1} a_m \sum_{\substack{L<\ell\le L_1\\ m\ell\equiv -2(d)}} e(\alpha m^2\ell^2k)$$

and

$$W_1' = \sum_{d\le D}\lambda(d) \sum_{0<|k|\le K} c(k)e(\beta k) \sum_{M<m\le M_1} a(m) \sum_{\substack{L<\ell\le L_1\\ m\ell\equiv -2(d)}} \log \ell e(\alpha m^2\ell^2k),$$

where

$$M_1\le 2M, \quad L_1\le 2L, \quad ML\asymp x, \quad L\ge w, \quad a(m)\ll x^{\varepsilon}. \tag{41}$$

The sums of the second type are

$$W_2 = \sum_{d\le D}\lambda(d) \sum_{0<|k|\le K} c(k)e(\beta k) \sum_{L<\ell\le L_1} b(\ell) \sum_{\substack{M<m\le M_1\\ m\ell\equiv -2(d)}} a(m)e(\alpha m^2\ell^2k),$$

where

$$M_1\le 2M, \quad L_1\le 2L, \quad ML\asymp x, \quad u\le L\le v, \quad a(m),\ b(\ell)\ll x^{\varepsilon}. \tag{42}$$

5.2.1. Evaluation of type II sums. Applying the Cauchy–Schwarz inequality to W_2 and using Lemma 2(i), (40), (42) and (46) we obtain that

$$W_2^2 \ll x^\varepsilon KDM \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,\,a)=1}} \sum_{\substack{\ell_i\sim L\\ i=1,2\\ \ell_1\equiv\ell_2(d)}} b(\ell_1)b(\ell_2) \sum_{\substack{m\sim M\\ m\ell_i\equiv a(d)\\ i=1,2}} e(\alpha m^2(\ell_1^2-\ell_2^2)k) \tag{43}$$

$$= W_{21} + x^{1+\varepsilon}MDK^2$$

where

$$W_{21} = x^\varepsilon KDM \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,\,a)=1}} \sum_{\substack{\ell_i\sim L\\ \ell_1\neq\ell_2\\ \ell_1\equiv\ell_2(d)}} b(\ell_1)b(\ell_2) \sum_{\substack{m\sim M\\ m\ell_i\equiv a(d)\\ i=1,2}} e(\alpha m^2(\ell_1^2-\ell_2^2)k)$$

Applying again the Cauchy–Schwarz inequality and substituting

$$m_1 = m_2 + td, \quad t \ll \frac{M}{D} \quad \text{and} \quad \ell_1 = \ell_2 + \omega d, \quad \omega \ll \frac{L}{D}$$

we sequentially obtain

$$W_{21}^2 \ll x^{2+\varepsilon}D^2K^3 \sum_{k\sim K} \sum_{\substack{d\sim D\\(d,\,a)=1}} \sum_{\substack{\ell_i\sim L\\ \ell_1\neq\ell_2\\ \ell_1\equiv\ell_2(d)}} \sum_{\substack{m_i\sim M\\ m_1\ell_i\equiv a(d)\\ m_2\ell_i\equiv a(d)\\ m_1\neq m_2}} e(\alpha(m_1^2-m^2)(\ell_1^2-\ell_2^2)k) \tag{44}$$

$$+ x^{3+\varepsilon}LDK^4$$

$$= W_{22} + x^{3+\varepsilon}LDK^4$$

with

$$W_{22} \ll x^{2+\varepsilon}D^2K^3 \sum_{k\sim K} \sum_{d\sim D} \sum_{\omega\ll\frac{L}{D}} \sum_{\ell_2\sim L} \sum_{t\ll\frac{M}{D}} \min\left\{\frac{M}{d},\ \frac{1}{\|\alpha d^3 t\omega(2\ell_2+\omega d)k\|}.\right\}$$

Putting $\ell = \ell_2 + \omega d$ and $z = \omega\ell k t$ we get

$$W_{22} \ll x^{2+\varepsilon}D^2K^3 \sum_{d\sim D} \sum_{z\ll\frac{xLK}{D^2}} \tau_4(z) \min\left\{\frac{x^2K}{d^3z},\ \frac{1}{\|\alpha d^3 z\|}.\right\} \tag{45}$$

If

$$\Delta < D \leq x^{\frac{8}{27}}\Delta \tag{46}$$

then from inequality (10) of Lemma 7, (45), (43) and (44) we get

$$W_2 \ll x^\varepsilon\left(x^{\frac{71}{72}}\Delta^{\frac{33}{64}}K + \frac{xK}{q^{\frac{1}{16}}} + \frac{xK}{\Delta^{\frac{1}{16}}} + x^{\frac{15}{16}}q^{\frac{1}{32}}K^{\frac{31}{32}}\right). \tag{47}$$

If

$$D \leq \Delta \tag{48}$$

we will estimate the sum W_{22} by putting $u = d^3z$ and applying Lemma 4, Lemma 2 (ii) to find

$$W_2 \ll x^\varepsilon\left(x^{\frac{71}{72}}\Delta^{\frac{33}{64}}K + \frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{4}}} + \frac{xK}{q^{\frac{1}{16}}} + \frac{xK}{\Delta^{\frac{1}{16}}} + x^{\frac{15}{16}}q^{\frac{1}{32}}K^{\frac{31}{32}} + x^{\frac{1}{2}}\Delta^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{49}$$

From (47) and (50) we get

$$W_2 \ll x^{\varepsilon}\left(x^{\frac{71}{72}}\Delta^{\frac{33}{64}}K+\frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{4}}}+\frac{xK}{q^{\frac{1}{32}}}+\frac{xK}{\Delta^{\frac{1}{16}}}+x^{\frac{15}{16}}q^{\frac{1}{32}}K^{\frac{31}{32}}+x^{\frac{1}{2}}\Delta^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{50}$$

5.2.2. Evaluation of type I sums. Reasoning as in the estimation of the sum W_{13} (see (5.1.2)) we obtain

$$W_1^2 \ll x^{\varepsilon}MDK\sum_{z\sim MD}\tau(z)\sum_{t\ll\frac{LK}{D}}\tau_3(t)\min\left\{\frac{x^2K}{z^2t},\,\frac{1}{\|\alpha z^2t\|}\right\}+x^{\varepsilon}MDK^2. \tag{51}$$

Using Lemma 5 and $ML\sim x$ we obtain

$$W_1^2 \ll x^{\varepsilon}\left(MDK^2+\frac{x^2K^2}{(MD)^{\frac{1}{2}}}+\frac{x^2K^2}{q^{\frac{1}{2}}}+xK^{\frac{3}{2}}q^{\frac{1}{2}}\right) \tag{52}$$

Using the inequality (51) we will evaluate the sum W_1 in one more way. Let $u=z^2t$ and from Lemma 4 (iii) and Lemma 2 (iv) we find

$$\begin{aligned} W_1^2 &\ll x^{\varepsilon}MDK\sum_{u\ll xMDK}\tau_5(u)\min\left\{\frac{x^2K}{u},\,\frac{1}{\|\alpha u\|}\right\} \\ &\ll x^{\varepsilon}\left(MDKq+xM^2D^2K^2+\frac{x^2MDK^2}{q}\right) \end{aligned} \tag{53}$$

If $MD>\Delta$ using the estimate (52) we get

$$W_1 \ll x^{\varepsilon}\left(x^{\frac{47}{48}}\Delta^{\frac{63}{128}}K+\frac{xK}{\Delta^{\frac{1}{4}}}+\frac{xK}{q^{\frac{1}{4}}}+x^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{54}$$

If $MD\le\Delta$ from (53) it follows

$$W_1 \ll x^{\varepsilon}\left(x^{\frac{1}{2}}\Delta K+\frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{2}}}+\Delta^{\frac{1}{2}}K^{\frac{1}{2}}q^{\frac{1}{2}}\right), \tag{55}$$

then using (54) and (55) we get

$$W_1 \ll x^{\varepsilon}\left(x^{\frac{47}{48}}\Delta^{\frac{63}{128}}K+\frac{xK}{\Delta^{\frac{1}{4}}}+\frac{xK}{q^{\frac{1}{4}}}+\frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{2}}}+x^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{56}$$

From (39), (50) and (56)

$$W \ll x^{\varepsilon}\left(\frac{xK}{\Delta^{\frac{1}{32}}}+\frac{xK}{q^{\frac{1}{32}}}+\frac{x\Delta^{\frac{1}{2}}K}{q^{\frac{1}{4}}}+x^{\frac{71}{72}}\Delta^{\frac{33}{64}}K+x^{\frac{15}{16}}K^{\frac{31}{32}}q^{\frac{1}{32}}+x^{\frac{1}{2}}\Delta^{\frac{1}{2}}K^{\frac{3}{4}}q^{\frac{1}{4}}\right). \tag{57}$$

5.3. Proof of Lemma 1. In Theorem 2 choose

$$x=q,\quad \Delta=K^{\frac{32.34}{33}},\quad K=x^{\frac{1}{1296}-\eta},\quad D=\frac{x^{1/2}}{\Delta K^4}, \tag{58}$$

where η is arbitrary small fixed number.

6. Proof of Theorem 2. As in [13] we take a periodic function with period 1 such that

$$\begin{aligned} 0<\chi(t)<1 \quad &\text{if} \quad -\delta<t<\delta; \\ \chi(t)=0 \quad &\text{if} \quad \delta\le t\le 1-\delta, \end{aligned} \tag{59}$$

and which has a Fourier series

$$\chi(t) = \delta + \sum_{|k|>0} c(k)e(kt), \tag{60}$$

with coefficients satisfying

$$\begin{aligned} c(0) &= \delta, \\ c(k) &\ll \delta \qquad \text{for all } k, \\ \sum_{|k|>K} |c(k)| &\ll x^{-1} \end{aligned} \tag{61}$$

and δ and K satisfying the conditions (8).

The existence of such a function is a consequence of a well known lemma of Vinogradov (see [9], ch. 1, §2).

Next we will use sieve methods. As usual, for any sequence $\mathcal{A}$ of integers weighted by the numbers f_n, $n \in \mathcal{A}$, we set

$$S(\mathcal{A}, z) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} f_n$$

and denote by $\mathcal{A}_d$ be the subsequence of elements $n \in \mathcal{A}$ with $n \equiv 0 \pmod d$. We write

$$P(z) = \prod_{p<z} p, \quad V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right) \quad \text{and} \quad C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

and we will use the linear sieve due to Iwaniec – this is Theorem 3 (see [7]).

To prove Theorem 2, it suffice to show that

$$S(\mathcal{A}, N^{1/3}) = \sum_{\substack{n+2 \le x \\ (n+2, P(x^{1/3})=1}} \chi(\alpha n^2 + \beta)\Lambda(n) > 0. \tag{62}$$

Following the exposition in Shi's article (see [11]) we have that

$$S \geq \sum_{\substack{n+2 \le x \\ (n+2, P(x^{1/12})=1}} \chi(\alpha n^2 + \beta)\Lambda(n) \Bigg(1 - \frac{1}{2} \sum_{\substack{x^{1/12} \le p_1 < x^{1/3.1} \\ n \equiv -2 (p_1)}} 1 - \frac{1}{2} \sum_{\substack{n+2 = p_1 p_2 p_3 \\ x^{1/12} \le p_1 < x^{1/3.1} \\ x^{1/3.1} \le p_2 < (\frac{x}{p_1})^{1/2}}} 1$$

$$- \sum_{\substack{n+2 = p_1 p_2 p_3 \\ x^{1/12} \le p_1 < p_2 < (\frac{x}{p_1})^{1/2}}} 1 + O\big(x^{11/12}\big) \Bigg).$$

So

$$S \geq S(\mathcal{A},\, x^{1/12}) - \frac{1}{2} \sum_{x^{1/12} \leq p_1 < x^{1/3.1}} S(\mathcal{A}_{p_1},\, x^{1/12}) - \frac{1}{2} \sum_{\substack{x^{1/12} \leq p_1 < x^{1/3.1} \\ x^{1/3.1} \leq p_2 < (\frac{x}{p_1})^{1/2}}} S(\mathcal{A}_{p_1 p_2},\, x^{1/12})$$

$$- \sum_{x^{1/12} \leq p_1 < p_2 < (\frac{x}{p_1})^{1/2}} S(\mathcal{A}_{p_1 p_2},\, x^{1/12}) + O\big(x^{11/12}\big)$$

$$= S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 + O\big(x^{11/12}\big)$$

and it is enough to proof that above expression is positive. Consider a square-free number d. If $2|d$, then we write $|\mathcal{A}_d| = |r(\mathcal{A}, d)| \leq 1$. Otherwise we have by the Fourier expansion of $\chi(n)$ that

$$|\mathcal{A}_d| = \sum_{\substack{n \leq x-2 \\ n \equiv -1(d)}} \chi(\alpha n^2 + \beta)\Lambda(n)$$

$$= \sum_{\substack{n \leq x-2 \\ n \equiv -1(d)}} \Bigg(\delta + \delta \sum_{0<|k|<K} c(k) e(\alpha n^2 k)\Lambda(n) + O(x^{-1})\Bigg)$$

$$= \delta\Bigg(\frac{x}{\varphi(d)} + R_1(d) + R_2(d) + O\bigg(\frac{x}{d(\log x)^A}\bigg)\Bigg).$$

Here $c(k) \ll 1$,

$$R_1(d) = \sum_{\substack{p \leq x-2 \\ p \equiv -1(d)}} 1 - \frac{x}{\varphi(d)}$$

$$R_2(d) = \sum_{0<|k|<K} c(k) \sum_{\substack{n \leq x-2 \\ n \equiv -2(d)}} e(\alpha n^2 k)\Lambda(n)$$

Applying Bombieri–Vinogradov theorem (see [8], Theorem 17.1)

$$\sum_{d \leq D} |R_1(d)| \ll \frac{x}{(\log x)^A}.$$

On the other hand , Theorem 1 implies that for a well-separated numbers d of level $D = \dfrac{x^{1/2}}{\Delta K^4}$ and $\lambda(d) \ll \tau(d)$ we get

$$\sum_{d \leq D} \lambda(d) R_2(d) \ll \frac{x}{(\log x)^A}$$

when $q = x$, where a/q convergent to α with a large enough denominator. From here on, the reasoning we go through is the same as in Shi's paper (see [11]). We will only note that to estimate the sum

$$\sum_{x^{1/12}\le p<x^{1/3.1}} \sum_{d\le D} R_2(pd)$$

with $D=\dfrac{x^{1/2}}{\Delta p K^4}$ first we present it as a $O(\log^4 x)$ number of sums of the type

$$R_2(P)=\sum_{d\sim D}\lambda(d)\sum_{1\le|k|\sim K}c(k)\sum_{p\sim P}\sum_{\substack{n\sim x\\ n+2\equiv 0\,(d)\\ n+2\equiv 0\,(p_1)}} e\big(\alpha n^2 k\big)\Lambda(n),$$

where $x^{1/12}\le P<x^{1/3.1}/2$.

If $DP\le x^{8/27}\Delta$ we put $t=dp$ and represent the sum $R_2(P)$ in type:

$$R_2(P)=\sum_{1\le|k|\sim K}c(k)\sum_{t\sim DP}g(t,\,d)\sum_{\substack{n\sim x\\ n+2\equiv 0\,(t)}} e\big(\alpha n^2 k\big)\Lambda(n),$$

where

$$g(t,\,d)=\sum_{\substack{d\sim D\\ d|(t,\,P(z))\\ t/d>x^{1/12}\\ t/d-\text{prime}}}\lambda(d)\ll\tau(t)$$

and evaluation is in the same way as in § 5.2.

If $DP\ge x^{8/27}\Delta$ then, depending on which interval P falls into, and bearing in mind Remark 2 and the fact that d is well-separated, we choose H so that PH falls into one of the intervals $x^{2/5}\le PH\le\dfrac{x^{1/2}}{\Delta K^4}$ or $x^{8/27}\Delta\le PH\le x^{2/5}$. So

$$\begin{aligned}R_2(P)&=\sum_{1\le|k|\sim K}c(k)\sum_{s\sim S}\sum_{h\sim H}\lambda(hs)\sum_{p_1\sim P}\sum_{\substack{n\sim x\\ n+2\equiv 0\,(d)\\ n+2\equiv 0\,(p_1)}} e\big(\alpha n^2 k\big)\Lambda(n)\\ &=\sum_{1\le|k|\sim K}c(k)\sum_{s\sim S}\sum_{t\sim PH}g(t,\,s)\sum_{\substack{n\sim x\\ n+2\equiv 0\,(ts)}} e\big(\alpha n^2 k\big)\Lambda(n)\end{aligned}$$

where

$$g(t,\,s)=\sum_{\substack{h\sim H\\ h|(t,\,P(z))}}\lambda(hs)\ll\tau(t)$$

and evaluation is in the same way as in § 5.1.

Using the same calculation as in [11] with

$$z=x^{\frac{1}{12}},\quad \Delta=K^{\frac{32.34}{33}},\quad K=x^{\frac{1}{1296}-\eta},\qquad D=\frac{x^{1/2}}{\Delta K^4},$$

we get that inequality (62) is true and the proof of Theorem is complete.

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