МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2025
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2025
Proceedings of the Fifty-Fourth Spring Conference
of the Union of Bulgarian Mathematicians
Varna, March 31 – April 4, 2025

LAGRANGE MULTIPLIERS IN DYNAMIC OPTIMIZATION

Mikhail I. Krastanov[1], **Nadezhda K. Ribarska**[2]

[1] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Sofia, Bulgaria
[2] Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
Sofia, Bulgaria
e-mails: [1]krast@math.bas.bg, [2]ribarska@fmi.uni-sofia.bg

An approach for studying optimization problems in abstract setting is presented and a general Lagrange multiplier rule is formulated. It is described how to apply this result to the basic problem of calculus of variations with pure state constraints of equality-type, as well as to a Mayer optimal control problem with pure state constraints.
Keywords: infinite-dimensional problems, approximating cone, nonseparation of sets, Lagrange multiplier rule, Pontryagin maximum principle

МНОЖИТЕЛИ НА ЛАГРАНЖ В ДИНАМИЧНАТА ОПТИМИЗАЦИЯ

Михаил И. Кръстанов[1], **Надежда К. Рибарска**[2]

[1] Институт по математика и информатика, Българска академия на науките,
София, България
[2] Факултет по математика и информатика,
Софийски университет „Св. Климент Охридски", София, България
e-mails: [1]krast@math.bas.bg, [2]ribarska@fmi.uni-sofia.bg

Представен е подход за изследване на оптимизационните задачи в абстрактна постановка и е формулирано общо правило от тип на множители на Лагранж. Описано е как да се приложи този резултат към основната задача на вариационното смятане с чисти фазови ограничения от тип равенство, както и към задача на Майер за оптимално управление с чисти фазови ограничения.

`https://doi.org/10.55630/mem.2025.54.009-017`

* The study is financed by the Sofia University St. Kliment Ohridski under contracts No. 80-10-8/29.03.2024 and No. 80-10-138/17.04.2024.

2020 Mathematics Subject Classification: 49K27, 49K15, 46N10.

Ключови думи: безкрайномерни задачи, апроксимиращ конус, неразделяне на множества, принцип на Лагранж, принцип на максимума на Понтрягин

1. Introduction. We consider the following basic problem of calculus of variations

$$\int_a^b L(x(t), \dot{x}(t))dt \to \min,$$

under the pure state constraint

$$x(t) \in \Phi \text{ for all } t \in [a, b],$$

where Φ is a closed subset of $\boldsymbol{R}^n$. Moreover, the following terminal constraints are imposed: $x(a) = x_a$ and $x(b) = x_b$, where x_a and x_b are two fixed points of Φ.

It is well-known that the necessary optimality condition for this problem without state constraints and with smooth integrand is given by the classical Euler-Lagrange equation. Finding nontrivial necessary optimality conditions in the presence of state constraints is important and there is a huge amount of papers devoted to this problem. The theory of state-constrained optimal control problems (containing the considered problem) originates from the seminal monograph [19] and the papers of Gamkrelidze (see [11, 12]). These results concern inequality-type constraints as well as equality-type constraints. Usually, subsequent developments in this theory deal with inequality-type constraints. Some important results in this direction can be found in the works of Dubovitskii and Milyutin [10], Neustadt [18], Warga [23], and others. Necessary nonsmooth optimality conditions of Pontryagin maximum principle type for state constrained problems were proved by Warga [24], Vinter and Pappas [22], Clarke [6], and Ioffe [13]. The books of Vinter [21]) and Clarke [7] contain most of the achievements to date. To our knowledge, the most recent developments can be found in Ioffe's papers [14, 15].

Our approach is based on systematic application of the methods of functional analysis and of non-smooth analysis, considering the problems of calculus of variations and of optimal control as constrained optimization problems in infinite-dimensional spaces. As early as 1965, Dubovickiĭ and Miljutin [10] applied a functional analysis based approach. These ideas are further developed by Aubin and Frankowska [1] in much more general situations. They realized the importance of set-valued mappings for obtaining necessary conditions for nonlinear problems in optimization, in particular in dynamic optimization. These ideas have proven their effectiveness over the years.

To formulate the problem written above as an optimization problem on an infinite-dimensional space, we consider

$$\varphi(x, y) \to \min \quad \text{subject to} \quad (x, y) \in \tilde{S} \cap (H \times Y),$$

where

$$\varphi(x, y) := \int_a^b L(x(t), y(t))dt,$$

$$\tilde{S} := \left\{ (x, y) \in X \times Y : \ x(t) := x_a + \int_a^t y(\tau)d\tau, \ x(b) = x_b \right\},$$

$$H := \{x \in X : \ x(t) \in \Phi\}, \text{ with } \Phi \text{ a closed subset of } \boldsymbol{R}^n,$$

$$x \in X := L^\infty([a, b]; \boldsymbol{R}^n) \text{ and } y \in Y := L^1([a, b]; \boldsymbol{R}^n).$$

So, the original problem is reduced to an optimization problem in infinite-dimensional state space. This problem has specific features which will be taken into account.

The survey is organized as follows: in the next section, we present a general approach to optimization problems in infinite-dimensional state space. In Section 3, a Lagrange multiplier rule based on uniform approximations is applied to the basic problem of calculus of variations with pure state constraints of equality-type. The last section is devoted to an application of our approach to a Mayer optimal control problem with pure state constraints. The main assumption is a variational condition involving a measure of noncompactness.

2. A general approach to optimization problems in infinite-dimensional state space. The term subtransversality is introduced in [9] in relation to proving linear convergence of the alternating projections algorithm. However, this property has been around for more than 20 years but under different names, see [17, Remark 4] and references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms, and a qualification condition for the normal intersection property with respect to the limiting normal cones and a sum rule for the limiting subdifferentials. The precise definition is:

Definition 2.1. *Let A and B be closed subsets of the Banach space X. Then A and B are said to be subtransversal at $x_0 \in A \cap B$ if there exist $\delta > 0$ and $K > 0$ such that*

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B)),$$

for all $x \in x_0 + \delta \bar{\mathbf{B}}$.

From our point of view, the most remarkable thing about subtransversality is the fact that it implies a rather general nonseparation result and respective Lagrange multiplier rule. The following result is taken from [2].

Theorem 2.2 (Lagrange multiplier rule). *Let us consider the optimization problem,*

$$f(x) \to \min \qquad \text{subject to} \qquad x \in S,$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper, and S is a closed subset of the Banach space X. Let $\bar{x}$ be a solution of the above problem, and $\tilde{C}_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))$ and $C_S(\bar{x})$ be closed convex cones contained in the corresponding Bouligand approximating cones $T_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))$ and $T_S(\bar{x})$. We assume that at least one of them consists of derivable tangent vectors.

(a) If $\tilde{C}_{\mathrm{epi} f}(\bar{x}, f(\bar{x})) - C_S(\bar{x}) \times (-\infty, 0]$ is not dense in $X \times \mathbb{R}$, then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

- *(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;*
- *(ii) $\eta \in \{0, 1\}$;*
- *(iii) $\langle \xi, v \rangle \leq 0$, for every $v \in C_S(\bar{x})$;*
- *(iv) $\langle \xi, w \rangle + \eta s \geq 0$, for every $(w, s) \in \tilde{C}_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))$.*

(b) If $\tilde{C}_{\mathrm{epi} f}(\bar{x}, f(\bar{x})) - C_S(\bar{x}) \times (-\infty, 0]$ is dense in $X \times \mathbb{R}$, then epi f *and $S \times (-\infty, f(\bar{x})]$ are not subtransversal at $(\bar{x}, f(\bar{x}))$.*

According to the above theorem, in order to construct a Lagrange multiplier, it is sufficient to prove that under certain conditions the case (b) is impossible. Two sufficient conditions for subtransversality in specific situations will be presented below.

3. A Lagrange multiplier rule based on uniform approximations. The following definition is taken from [16].

Definition 3.1. *Let S be a closed subset of X, and x_0 belong to S. We say that the bounded set D is* ***a uniform tangent set*** *to S at the point x_0 if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $v \in D$ and $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$, one can find $\lambda > 0$ such that $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for every $t \in [0, \lambda]$.*

The next theorem is the main result from [4].

Theorem 3.2. *Let S be a closed subset of X, and x_0 belong to S. The following are equivalent:*

1. *D is a uniform tangent set to S at the point x_0;*
2. *for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $v \in D$ and $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$, one can find a sequence of positive reals $t_m \to 0$ such that $S \cap (x + t_m(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for every positive integer m;*
3. *for every $\varepsilon > 0$, there exist $\delta > 0$ and $\lambda > 0$ such that, for all $v \in D$ and $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$, the set $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for every $t \in [0, \lambda]$.*

The basic properties of uniform tangent sets are gathered in the next proposition.

Proposition 3.3. *Let S be a closed subset of X, and $x_0 \in S$. Let D be a uniform tangent set to S at the point x_0. Then the following hold true:*

1. *if $c > 0$ is fixed, then cD is a uniform tangent set to S at x_0;*
2. *if $D' \subset D$, then D' is a uniform tangent set to S at x_0;*
3. *if D' is another uniform tangent set to S at x_0, then $D \cup D'$ is a uniform tangent set to S at x_0;*
4. *the closure of D is a uniform tangent set to S at x_0;*
5. *$\overline{\mathrm{co}}\, D$ is a uniform tangent set to S at x_0;*
6. *if S is convex, then $(S - x_0) \cap M\overline{\mathbf{B}}$ is a uniform tangent set to S at x_0, for every $M > 0$.*

We are going to use the original definition of the Clarke tangent cone in Banach spaces (see, e.g., [20]).

Definition 3.4. *Let S be a closed subset of X, and x_0 belong to S. We say that $v \in X$ is* ***a tangent vector*** *to S at the point x_0 if for every $\varepsilon > 0$, there exist $\delta > 0$ and $\lambda > 0$ such that, for all $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$, the set $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for every $t \in [0, \lambda]$. The set of all tangent vectors to S at x_0 is called* ***Clarke tangent cone*** *and is denoted by $\widehat{T}_S(x_0)$.*

Theorem 3.2 implies that every uniform tangent set to S at x_0 is contained in $\widehat{T}_S(x_0)$, provided S is a closed subset of a Banach space.

Definition 3.5. *Let Z be a Banach space, and $S \subset Z$. The set S is said to be quasisolid if its closed convex hull $\overline{\mathrm{co}}\, S$ has a nonempty interior in its closed affine hull, i.e. if there exists a point $z_0 \in \mathrm{co}\, S$ such that $\overline{\mathrm{co}}\,(S - z_0)$ has a nonempty interior in $\overline{\mathrm{span}}(S - z_0)$, the closed subspace spanned by $(S - z_0)$.*

The following Lagrange multiplier rule is proved in [4]. This assertion can be obtained as a corollary of Theorem 2.2 by taking into account Propositions 3.4 and 3.16 from [3].

Theorem 3.6. *Let us consider the optimization problem,*

$$f(x) \to \min \quad \text{subject to} \quad x \in S,$$

where X is a Banach space, $f : X \to \boldsymbol{R} \cup \{+\infty\}$ is a proper lower semicontinuous function, and S is a closed subset of X. Let $\bar{x}$ be a solution of this problem, D_S a uniform tangent set to S at the point $\bar{x}$, and $D_{\mathrm{epi}f}$ a uniform tangent set to epi f *at $(\bar{x}, f(\bar{x}))$. We assume that the set $D_S \times [-1, 0] - D_{\mathrm{epi}f}$ is quasisolid. Then there exists a* *pair* $(\xi, \eta) \in X^* \times \boldsymbol{R}$ *such that*

i/ $(\xi, \eta) \neq (\mathbf{0}, 0)$*;*

ii/ $\eta \in \{0, 1\}$*;*

iii/ $\langle \xi, v \rangle \leq 0$*, for every* $v \in D_S$*;*

iv/ $\langle \xi, w \rangle + \eta r \geq 0$*, for every* $(w, r) \in D_{\text{epi} f}$.

To apply this theorem, we consider the following basic problem of calculus of variations,

$$\int_a^b L(x(t), \dot{x}(t))dt \to \min,$$

under the pure state constraints

$$x(t) \in \Phi \text{ for all } t \in [a, b],$$

where Φ is a closed subset of $\boldsymbol{R}^n$, L is Lipschitz continuous with constant Q, and x is absolutely continuous with $\dot{x} \in L^1([a, b]; \boldsymbol{R}^n)$. Moreover, the following terminal constraints are imposed: $x(a) = x_a$ and $x(b) = x_b$, where x_a and x_b are two fixed points of Φ.

Let us recall that

$$H := \{x \in L^\infty([a, b]; \boldsymbol{R}^n) : \ x(t) \in \Phi\}.$$

We formulate a necessary optimality condition under the following assumptions:

Assumption A1. For every compact set K of the set Φ, there exists an open neighborhood Ω of the set K, an open convex neighborhood ω of the origin in $\boldsymbol{R}^n$, and a continuously differentiable map $\psi : U \to \boldsymbol{R}^n$, where $U := \{(x, w) : w \in \omega, x \in \Omega\}$, such that the following properties hold true:

i/ $\psi(x, 0) = x$, for every $x \in \Omega$;

ii/ $\psi'(x, 0)(0, w) = w$, for every $x \in \Omega$ and $w \in \omega$;

iii/ whenever $x \in \Omega \cap \Phi$ and $w \in \hat{T}_\Phi(x) \cap \omega$, the point $\psi(x, w)$ is in Φ.

Assumption A2. Let $D_H(\bar{x})$ be a uniform tangent set to H at $\bar{x}$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(\bar{u}, \bar{v}) \in S \cap (D_H(\bar{x}) \times \bar{\mathbf{B}}_Y)$ and $(x, y) \in ((\bar{x}, \bar{y}) + \delta \bar{\mathbf{B}}_{X \times Y}) \cap B$, there exists $(u, v) \in ((\bar{u}, \bar{v}) + \varepsilon \bar{\mathbf{B}}_{X \times Y}) \cap S \cap (\hat{T}_H(x) \times Y)$.

Theorem 3.7. *Let* $(\bar{x}, \dot{\bar{x}})$ *be a local (in* $L^\infty([a, b]; \boldsymbol{R}^n) \times L^1([a, b]; \boldsymbol{R}^n)$*) solution of the above written problem, and* $D_H(\bar{x})$ *a uniform tangent set to* H *at the point* $\bar{x}$*. Let Assumptions A1 and A2 be fulfilled. Then there exists* $(\xi, \eta) \in L^1([a, b], \boldsymbol{R}^n) \times L^\infty([a, b], \boldsymbol{R}^n)$ *such that* $(\xi(t), \eta(t)) \in \partial L(x(t), y(t))$ *and*

$$\frac{d}{dt}\langle \eta(t), u(t) \rangle \geq \langle \xi(t), u(t) \rangle + \langle \eta(t), \dot{u}(t) \rangle,$$

for every absolutely continuous $u \in D_H(\bar{x})$*, almost everywhere in* $[a, b]$.

As a corollary of this result, we obtain in the following assertions.

Corollary 3.8. *Let* $(\bar{x}, \dot{\bar{x}})$ *be a local (in* $L^\infty([a, b]; \boldsymbol{R}^n) \times L^1([a, b]; \boldsymbol{R}^n)$*) solution of the above written problem, and* $D_H(\bar{x})$ *a uniform tangent set to* H *at the point* $\bar{x}$ *which is symmetric, i.e.* $D_H(\bar{x}) = -D_H(\bar{x})$*. Let Assumptions A1 and A2 be fulfilled. Then there exists* $(\xi, \eta) \in L^1([a, b], \boldsymbol{R}^n) \times L^\infty([a, b], \boldsymbol{R}^n)$ *such that* $(\xi(t), \eta(t)) \in \partial L(x(t), y(t))$ *and, for every absolutely continuous* $u \in D_H(\bar{x})$*, the function* $\langle \eta, u \rangle$ *is absolutely continuous and, for almost all* $t \in [a, b]$*, we have*

$$\frac{d}{dt}\langle \eta(t), u(t) \rangle = \langle \xi(t), u(t) \rangle + \langle \eta(t), \dot{u}(t) \rangle.$$

Corollary 3.9. *Let* Φ *be a* C^2*-smooth manifold, and* $(\bar{x}, \dot{\bar{x}})$ *a local (in* $L^\infty([a, b]; \boldsymbol{R}^n) \times$

$L^1([a,b];\boldsymbol{R}^n)$) *solution of the above written problem. Then there exists a pair* $(\xi,\eta) \in L^1([a,b],\boldsymbol{R}^n) \times L^\infty([a,b],\boldsymbol{R}^n)$ *such that* $(\xi(t),\eta(t)) \in \partial L(x(t),y(t))$ *and the tangent component* η_τ *of* η *is differentiable for almost all* $t \in [a,b]$. *Moreover,*

$$\big(\xi(t) - \dot{\eta}_\tau(t), \eta(t) - \eta_\tau(t)\big)$$

is orthogonal to the tangent space $T_{(\bar{x}(t),\dot{\bar{x}}(t))}T\Phi$ *to the tangent bundle* $T\Phi$ *at the point* $(\bar{x}(t),\dot{\bar{x}}(t))$ *almost everywhere.*

4. A Lagrange multiplier rule based on a variational condition. The starting point for the next definition is the classical idea of "perturbation" and subsequent "correction" of the epigraph of the considered functional, introduced by J.M. Borwein and H.M. Strojwas in [5].

Definition 4.1. *Let* X *and* Y *be Banach spaces, and* $\varphi : X \times Y \to R \cup \{+\infty\}$ *a proper lower semicontinuous function which has finite value at* $(\bar{x},\bar{y}) \in X \times Y$. *It is said that* φ *satisfies the* ***variational condition*** *at* $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ *with correcting set* $U \subset X \times Y \times \boldsymbol{R}$ *if there exists a positive real* $\bar{\delta} > 0$ *such that, for every* $\lambda \in [0,\bar{\delta}]$, *the following inclusion holds true:*

$$\operatorname{epi}\varphi \cap \big((\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \bar{\delta}\cdot \bar{B}_{X\times Y\times R}\big) + \lambda\left(\bar{B}_X,0,0\right) \subset \operatorname{epi}\varphi + \lambda U.$$

Of course, this condition could be trivial if there were no restrictions put on the "correcting set". The following definition is well-known (see, e.g., [8, 7.3]).

Definition 4.2. *Let* X *be a normed space, and* A *a bounded subset of* X. *The nonnegative real*

$$\mu(A) := \inf\left\{ r > 0 : \text{there exist finitely many balls } B_i \text{ of radius } r \text{ with } A \subset \bigcup_{i=1}^k B_i \right\}$$

is called the ***ball measure of noncompactness*** *of the set* A.

Let X and Y be Banach spaces, $\varphi : X \times Y \to \boldsymbol{R} \cup \{+\infty\}$ a lower semicontinuous function, and $L : Y \to X$ a bounded linear operator whose graph is given by

$$S := \big\{(Ly,y) \in X \times Y : \ y \in Y\big\}.$$

Theorem 4.3. *Let* φ *satisfy the variational condition, where the "correcting set"* $U \subset X \times Y \times \boldsymbol{R}$, *having the form* $U = (U_X, U_Y, U_{\boldsymbol{R}})$, *is bounded and such that*

$$\mu\left(U_X - L(U_Y)\right) < 1,$$

where μ *denotes the ball measure of noncompactness (in* X*). Moreover, we assume that the cone* $\hat{T}_{\operatorname{epi}\varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) - S \times (-\infty,0]$ *is dense in* $X \times Y \times \boldsymbol{R}$. *Then the sets* epi φ *and* $(S + (\bar{x},\bar{y})) \times (-\infty, \varphi(\bar{x},\bar{y})]$ *are tangentially transversal at* $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$.

Consider the optimization problem,

$$\text{(OP)} \qquad \varphi(x,y) \to \min \quad \text{subject to} \quad (x,y) \in \widetilde{S} := (x_0,y_0) + S.$$

Theorem 4.4 (**Noncompactness theorem**). *In the above setting, let* $(\bar{x},\bar{y})$ *be a solution of* (OP), *and the functional* φ *have finite value at* $(\bar{x},\bar{y})$ *and satisfy the variational condition with a correcting set* U, *having the form* $U = (U_X, U_Y, U_{\boldsymbol{R}})$. *Let* U *be bounded and* $\mu(U_X - L(U_Y)) < 1$, *where* μ *denotes the ball measure of noncompactness (in* X*). Then there exists a triple* $(\xi,\mu,\zeta) \in X^* \times Y^* \times \mathbb{R}$ *such that*

(i) $(\xi,\mu,\zeta) \neq (\mathbf{0},\mathbf{0},0)$;

(ii) $\zeta \in \{0,1\}$;

(iii) $\langle \xi,u\rangle + \langle \mu,v\rangle = 0$, *for every* $(u,v) \in S$;

(iv) $\langle \xi,u\rangle + \langle \mu,v\rangle + \zeta w \geq 0$, *for every* $(u,v,w) \in \widehat{T}_{\operatorname{epi}\varphi}(\bar{x},\bar{y},\varphi((\bar{x},\bar{y}))$.

To prove the above theorem, it is sufficient to apply Theorem 2.2 with the Banach space being $X \times Y$, the functional being φ, and the constraint being $S+(x_0, y_0)$. Moreover, let the approximating cone to the epigraph $\operatorname{epi}\varphi$ at $(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$ be the corresponding Clarke tangent cone, $\widehat{T}_{\operatorname{epi}\varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$. Note that the Clarke (and Bouligand) tangent cone to the constraint is S. Then Theorem 2.2 is applicable and case (b) is impossible due to Theorem 4.3. The equality in Theorem 4.4(iii) is due to the fact that S is a linear space.

Let us make things less abstract. We are going to apply the Noncompactness theorem to the Banach spaces $X := L^\infty([a,b]; \boldsymbol{R}^n)$ (equipped with the usual essential supremum norm) and $Y := L^1([a,b]; \boldsymbol{R}^n)$ (equipped with the usual L^1-norm), and to the integration operator $L : Y \to X$ defined by

$$(Ly)(t) = \int_a^t y(s)ds, \quad t \in [a,b].$$

Proposition 4.5. *The linear operator $L : Y \to X$ is bounded of norm one. Moreover, if $U \subset Y$ is a weakly compact set, then $L(U)$ is totally bounded.*

As an immediate implication of the above proposition, we obtain a sufficient condition for a correcting set U which has the form $U = U_X \times U_Y \times U_{\boldsymbol{R}}$, where $U_X \subset X$, $U_Y \subset Y$, $U_{\boldsymbol{R}} \subset \boldsymbol{R}$ satisfy $\mu(U_X - L(U_Y)) < 1$. The sufficient condition is $U_X \subset M_X + r_X \bar{B}_X$ and $U_Y \subset M_Y + r_Y \bar{B}_Y$, where $M_X \subset X$ is norm compact, $M_Y \subset Y$ is weakly compact, and $r_X + r_Y < 1$. Indeed, then we have

$$\begin{aligned}\mu(U_X - L(U_Y)) &\le \mu(M_X + r_X \bar{B}_X) + \mu\big(L(M_Y) + r_Y L(\bar{B}_Y)\big) \\ &\le \mu(M_X) + r_X + \mu(L(M_Y)) + r_Y \mu\big(L(\bar{B}_Y)\big) \le r_X + r_Y < 1.\end{aligned}$$

Let us consider a process whose dynamics is determined by the following differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \text{ almost everywhere on } [a,b], \; x(a) = \mathbf{0}, \tag{1}$$

where the multivalued map $F : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^n$ has nonempty compact values and is integrally Lipschitz continuous, that is, there exists a summable nonnegative function k on $[a,b]$ with

$$F(t, x') \subset F(t, x) + k(t)\|x' - x\|\mathbb{B}, \quad \text{for each } x', x \text{ from } \mathbb{R}^n,$$

and for almost all $t \in [a,b]$. Then the set of solutions of this differential inclusion starting from the origin can be represented as $S \cap G$, where

$$G := \big\{(x,y) \in X \times Y : (x(t), y(t)) \in \operatorname{gr} F \text{ almost everywhere on } [a,b]\big\}.$$

Let H be a closed subset of X. For example, if Φ is a fixed closed subset of $\boldsymbol{R}^n$, then one can define the set H as

$$H := \{x \in X : \; x(t) \in \Phi \text{ for almost all } t \in [a,b]\}.$$

Let $l : \boldsymbol{R}^n \to \boldsymbol{R}$ be a Lipschitz continuous function, and δ_b a norm one extension to X of the Dirac functional on the space of continuous functions, i.e. $\delta_b(x) = x(b)$ for every $x \in C([a,b], \boldsymbol{R}^n)$. Next, we define $\varphi : X \to \boldsymbol{R}$ as $\varphi(x) := l(\delta_b(x))$, and consider the optimization problem,

$$l(\delta_b(x)) \to \min,$$

under the pure state constraints

$$x(t) \in \Phi \text{ for all } t \in [a,b],$$

where x is a solution of the differential inclusion (1). Using the notations written above, this problem can be stated as follows,

(OCP) $\varphi(x) \to \min$ subject to $(x,y) \in G \cap (H \times Y) \cap S.$

We set

$$\psi(x,y) := \begin{cases} \varphi(x), & \text{if } (x,y) \in (H \times Y) \cap G; \\ \infty, & \text{if } (x,y) \notin (H \times Y) \cap G. \end{cases}$$

Theorem 4.6. *In the above setting, let* $(\bar{x}, \bar{y})$ *be a solution to* (OCP)*,* H *be massive at* $\bar{x}$*, and the Clarke tangent cone* $\widehat{T}_H(\bar{x})$ *be finite codimentional. Then there exists a triple* $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$ *such that*

(i) $(\xi, \eta, \zeta) \neq (\mathbf{0}, \mathbf{0}, 0)$*;*

(ii) $\zeta \in \{0, 1\}$*;*

(iii) $\langle \xi, u \rangle + \langle \eta, v \rangle = 0$*, for every* $(u,v) \in S$*;*

(iv) $\langle \xi, u \rangle + \langle \eta, v \rangle + \zeta w \geq 0$*, for every* $(u,v,w) \in \widehat{T}_{\mathrm{epi}\psi}(\bar{x}, \dot{\bar{x}}, \varphi(\bar{x}))$*.*

To apply the above written theorem, we consider two possible cases:

1. Nondegenerate case: If $\hat{T}_G(\bar{x}, \dot{\bar{x}}) - \hat{T}_H(\bar{x}) \times Y$ is dense in $X \times Y$, then (because H and $\hat{T}_H(\bar{x})$ are massive) we obtain subtransversality of the sets G and $(H \times Y)$ at $(\bar{x}, \dot{\bar{x}})$, and of their tangent cones at the same point. Hence, $\hat{N}_G(\bar{x}, \dot{\bar{x}}) + \hat{N}_H(\bar{x}) \times Y \supset \hat{N}_{G \cap (H \times Y)}(\bar{x}, \dot{\bar{x}})$. Using the above inclusion, one can obtain that

$$(\xi, \eta, \zeta) = (\xi_G, \eta, 0) + (\xi_H, 0, 0) + (\langle \lambda, \delta_b \rangle, 0, \zeta),$$

where (ξ_G, η) is normal to G at $(\bar{x}, \dot{\bar{x}})$, ξ_H is normal to H at $\bar{x}$, and (λ, ζ) is normal to $\mathrm{epi}\, l$ at $\delta_b(\bar{x})$. Since (ξ, η) is normal to S, one can obtain from here the usual form of the Euler inclusion and of the transversality condition in the Pontryagin maximum principle.

2. Degenerate case: If $\hat{T}_G(\bar{x}, \dot{\bar{x}}) - \hat{T}_H(\bar{x}) \times Y$ is not dense in $X \times Y$, then there exists a nontrivial functional μ in X^* such that $(\mu, 0)$ separates $\hat{T}_G(\bar{x}, \dot{\bar{x}})$ and $\hat{T}_H(\bar{x}) \times Y$. Thus, $(\mu, 0)$ is normal to G at $(\bar{x}, \dot{\bar{x}})$, and $-\mu$ is normal to H at the point $\bar{x}$.

It is natural to ask when the above situation occurs. At least for pointwise constraint $H = \{x \in X : x(t) \in \Phi$ for almost all $t \in [a,b]\}$ and for autonomous differential inclusion, the answer is straightforward: the degenerate phenomenon takes place exactly when the following finite dimensional condition holds true,

$$\mathrm{int}\, (\hat{T}_H(\bar{x})) \cap \{u \in X : \exists v \in Y, (u,v) \in \hat{T}_G(\bar{x}, \dot{\bar{x}})\} \neq \emptyset.$$

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