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Embedding $\Lambda_\infty(\alpha)$ into $\Lambda_1(\alpha)$ and Some Consequences

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In this work we study the exponent sequences α for which $\Lambda_\infty(\alpha)$ is embeddable into $\Lambda_1(\alpha)$. We show that in this case α is necessarily weakly stable but weak stability is not sufficient in general. However we prove the following:

If $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ spanned by a block basic sequence then α is stable, the converse of which is well known. We also give equivalent conditions in terms of embeddings of power series spaces for weak stability of an exponent sequence.

1. Introduction and preliminaries

A matrix (a_{kn}) with i) $0 \leqq a_{kn} \leqq a_{k+1,n}$; ii) $\sup_k a_{kn} > 0$ for every n, k is called a Köthe matrix and the sequence space $E = K(a_{kn}) = \{(\xi_n) : \|(\xi_n)\|_k = \Sigma_n |\xi_n| \alpha_{kn} < +\infty \forall k\}$ topologized by the seminorms $(\|\cdot\|_k)$ is called a Köthe space. In the sequel we shall assume E is nuclear and has a continuous norm, and hence every (a_{kn}) will be assumed to satisfy 1) $0 < a_{kn} \leqq a_{k+1,n}$; 2) $\forall k \exists m (a_{kn}/a_{mn})_n \varepsilon l_1$. A Köthe space E is Fréchet (metrisable, complete lctvs**) and the sequence (e_n), $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with appearing at the n-th entry, forms a basis for E. A step space $E|_{(n_i)} = K(a_{kn_i})$ is the (closed) subspace of E generated by a subsequence (e_{n_i}) and step-spaces are always (topologically) complemented. We say E is of type d_i, $i = 0, 3, 5$, if E is generated by some (a_{kn}) with:

(d_0) For each k, $(a_{k+1,n}/a_{kn})_n$ is nondecreasing (in this case E is also called regular; if the ratio is strictly increasing than strictly regular).

(d_3) For each k, n $(a_{k+1,n})^2 \leqq a_{kn} a_{k+2,n}$

(d_5) E $M \geqq 1$ with $a_{k+1,n}/a_{kn} \leqq (a_{k+2,n}/a_{k+1,n})^M$ $\forall$ k, n.

A Köthe space E is called a G_∞-space if it is given by a Köthe matrix (a_{kn}) which satisfies 1) $a_{1n} = 1$ $\forall$ n; 2) $\forall$ k $\exists$ m with $\sup a_{kn}^2/a_{mn} < +\infty$; 3) $(a_{kn})_n$ is non-decreasing in n for each k. The special Köthe spaces $\Lambda_0(\alpha) = K\exp(-\alpha_n/k))$ and $\Lambda_\infty(\alpha) = K(\exp(k\alpha_n))$ are called power series spaces of finite resp.

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**locally convex topological vector space infinite type. Here $\alpha=(\alpha_n)$ is a sequence of positive real numbers increasing to ∞ rapidly enough so that the resulting space is nuclear. Such sequences will be called exponent sequences. The nuclearity condition is equivalent to: $\lim_n \log n/\alpha_n=0$ resp. $\sup_n \log n/\alpha_n<+\infty$ for finite resp. infinite types. We note also that replacing $(-1/k)$ resp. (k) by any (r_k) resp. (s_k) where r_k increases strictly to 0 and s_k to infinity does not alter the corresponding space or its topology. Moreover, if (r_k) increases strictly to some $r<+\infty$, then $\Lambda_r(\alpha)=K(\exp(r_k\alpha_n)) \cong \Lambda_0(\alpha)$ (topological isomorphism). Consequently, we shall use $\Lambda_1(\alpha)$ or $\Lambda_0(\alpha)$ without any further distinction. If α is equivalent to some exponent sequence β, denoted by $\alpha\sim\beta$, that is $\sup_n \alpha_n/\beta_n<+\infty$ and $\sup_n \beta_n/\alpha_n<+\infty$ then $\Lambda_t(\alpha)=\Lambda_t(\beta)$ $t=0,\infty$ and the topologies coincide. An exponent sequence α is called weakly stable if $\sup_n \alpha_{n+1}/\alpha_n<+\infty$ and stable if $\sup_n \alpha_{2n}/\alpha_n<+\infty$. It follows that α is weakly stable (resp. stable) if and only if $E\times K\cong E$ (resp. $E\times E \cong E$), where $E=\Lambda_t(\alpha)$, $t=0$ or $t=\infty$.

A sequence (x_n) in $E=K(a_{kn})$ is called a basic sequence if it forms a basis for its closed linear span. Any subsequence of (e_n) is a basic sequence. A more general class of basic sequences is given by $x_n=\Sigma_{i\in N_n} t_{in}\, e_i$, where t_{in} are scalars and (N_n) is a collection of pairwise disjoint, non-empty, finite subsets of N. A basic sequence of this form is called a block basic sequence.

If $E=K(b_{kn})$ is isomorphic to a (closed) subspace of $F=K(a_{kn})$ and if $Te_n=\Sigma_{i=1}^\infty t_{ni} f_i$ is an isomorphism $T\colon E\to F$ then, using the sup-norms, $\|x\|_k=\sup_n|x_n|a_{kn}$, if $x=\Sigma x_n e_n\in E$, which generate the original topology of E by nuclearity, we obtain the fundamental inequality (cf. [4]):

$$a_{jq_{kn}}/a_{kq_{kn}}\leqq b_{jn}/b_{kn}\leqq a_{jq_{jn}}/a_{kq_{jn}} \quad \forall\ k,\ j,\ n,$$

where $\|Te_n\|_k=\sup_i|t_{ni}|a_{ki}=|t_{nq_{kn}}|a_{kq_{kn}}$. Here obviously the supremum is attained at some indices; q_{kn} is taken to be the largest of such indices. It then follows that $q_{kn}\leqq q_{k+1,n}$ $\forall$ $n,\ k$ (cf. [4]). The next lemma serves as a partial converse, in that it tells us that we can construct sums where the q_{kn}'s can be controlled: (Cf. [4]).

Lemma 1.1. *Let (a_{kn}) be strictly regular, $q_1<q_2<\dots<q_m$ be natural numbers, $\{t_{q_k}: k=1, 2, \dots, m\}$ be real numbers with $t_{q_1}\neq 0$. Then if*

$$a_{k+1,q_k}/a_{k+1,q_{k+1}}\leqq|t_{q_{k+1}}|/|t_{q_k}|<a_{kq_k}/a_{kq_{k+1}}$$

holds for $k=1, 2, \dots, m-1$ then $\sup\{|t_{q_k}|a_{sq_k}: k=1, 2, \dots, m\}=|t_{q_s}|a_{sq_s}$ for $s=1, 2, \dots, m$.

For concepts, terminology and results not explained and as general references we refer the reader to [4], [5] or [6].

We state below some observations and characterisations which are simple consequences:

Lemma 1.2. *An exponent sequence α is weakly stable if and only if $\alpha\sim\beta$, where β is defined by $\beta_n=A^p$ if $r_p<n\leqq r_{p+1}$, for some $A>1$ and some subsequence (r_p) of natural numbers with $r_1=0$.*

Proof: Since sufficiency is trivial, suppose α is weakly stable, say $\sup_n \alpha_{n+1}/\alpha_n = A < +\infty$. Let $r_1=0$, $r_p=\max\{s: \alpha_s < A^p\}$ for $p>1$. Then since α is non-decreasing, (r_p) increases strictly. Define β by $\beta_n = A^p$ for $r_p < n \leqq r_{p+1}$, $p=1, 2, \ldots$. Then if $r_p < n \leqq r_{p+1}$ one has: $\alpha_n/\beta_n \leqq \alpha_{r_{p+1}}/A^p < A$ and $\beta_n/\alpha_n < A^p/\alpha_{r_p} \leqq A^{p+1}/\alpha_{r_p+1} \leqq A$ so $\alpha \sim \beta$.

Note 1.3. From now on, for a weakly stable exponent sequence α, a sequence β obtained from Lemma 1.2, will be called a standard form of α and we shall stick to the above notation, i. e. A and (r_p) will have the above meaning.

Lemma 1.4. *An exponent sequence α is stable if and only if it has a standard form with $r_{p+1} \geqq 2r_p$ $\forall$ p.*

Proof: Since sufficiency of the condition is obvious suppose α is stable, say $\sup_n \alpha_{2n}/\alpha_n = A < +\infty$. Let $r_p = \max\{s: \alpha_s < A^p\}$, $r_1=0$. Then $\alpha_{2r_p} \leqq A\alpha_{r_p} < A^{p+1} \leqq \alpha_{r_{p+1}+1}$ so $2r_p \leqq r_{p+1}$. The rest of the proof follows as in Lemma 1.2.

Lemma 1.5. *If a subsequence (α_{n_i}) of α is weakly stable, then α itself is also weakly stable.*

Proof: Immediate.

Remark 1.6. The conclusion of Lemma 1.5. is false if "weakly stable" is replaced by "stable". In fact it can be shown that a subsequence (α_{n_i}) of α is stable if and only if α has a standard form with $r_{p+1} \geqq r_p + 2^p$, which obviously does not imply the condition of Lemma 1.4.

2. Some embedding theorems

Proposition 2.1. *Let $E = \Lambda_\infty(\alpha)$ where α is weakly stable. Then any $F = K(a_{kn})$ of type d_3 has a step space which is isomorphic to a subspace of E.*

Proof: Suppose $\sup_n \alpha_{n+1}/\alpha_n = A < +\infty$. Let (b_{kn}) be a representation of F satisfying $(a_{k+1,n}/a_{kn})^A \leqq a_{k+2,n}/a_{k+1,n}$ for all k, n and $\lim_n a_{kn}/a_{k+1,n} = 0$ for all k (cf.[4]).

Claim: For any m_0, q_0 there exist $m > m_0$, $q_0 < q_{1m} < \ldots < q_{mm}$ satisfying

(1) $\exp(\alpha_{q_{km}}) \leqq b_{k+1,m}/b_{km} < \exp(\alpha_{q_{k+1,m}})$ $k=1, 2, \ldots, m-1$.

Proof of claim: Let m be the smallest index with $m > m_0$ and $\exp(\alpha_{q_0+1}) \leqq b_{2m}/b_{1m}$. To construct (q_{im}), $i=1, 2, \ldots, m$, inductively, let $q_{1m} = q_0 + 1$ and supposing $q_0 < q_{1m} < \ldots < q_{km}$ are defined so that (1) holds (q_{km} satisfies only the left inequality), let $q_{k+1,m}$ be the smallest index with $b_{k+1,m}/b_{km} < \exp(\alpha_{q_{k+1,m}})$. Then we have:

$$\exp(\alpha_{q_{k+1,m}}) \leqq \exp(A\alpha_{q_{k+1,m}-1}) \leqq (b_{k+1,m}/b_{km})^A \leqq b_{k+2,m}/b_{k+1,m},$$

hence the left inequality of (1) also holds for $k+1$. This completes the induction step and hence the proof of the claim. Now we construct two subsequences, (n_i) and $\{q_{1i}, \dots, q_{ii}\}$ for each i as follows: Let n_1 be such that $\exp(\alpha_2) \leqq b_{2n_1}/b_{1n_1}$ holds. Let $q_{11}=2$. Suppose n_i and $(q_{ki})_{k=1}^{i}$ are chosen. Using our claim with $m_0 = n_i$, $q_0 = q_{ii}$, we find indices $n_{i+1} > n_i$ and $q_{ii} < q_{1,i+1} < \dots < q_{i+1,i+1}$ so that (1) is satisfied.

We set $t_{q_{ki}} = b_{kn_i}/\exp(k\,\alpha_{q_{ki}})$ and define $y_i = \Sigma_{k=1}^{i} t_{q_{ki}} e_{q_{ki}}$ $i \in N$. Since q_{ki}'s all distinct, (y_i) is a block basic sequence in E and hence generates subspace of E. Now putting $b_{kn_i} = t_{q_{ki}} \exp(k\alpha_{q_{ki}})$ in (1) for each fixed i, we obtain

$$\exp((k+1)\alpha_{q_{ki}})/\exp((k+1)\alpha_{q_{k+1,i}}) \leqq t_{q_{k+1,i}}/t_{q_{ki}}$$

$$< \exp(k\alpha_{q_{ki}})/\exp(k\alpha_{q_{k+1,i}})$$

$k=1, 2, \dots, i$. It follows by Lemma 1.1. that

$$\| y_i \|_k = \sup\{ | t_{q_{mi}} \exp(k\alpha_{q_{mi}}) \colon m=1, 2, \dots, i\} = b_{kn_i}$$

$k=1, 2, \dots, i$. We conclude that $K(b_{kn_i})$ is isomorphic to a subspace of E.

Proposition 2.2. *Let $E=\Lambda_1(\alpha)$ where α is weakly stable. Then any $F=K(b_{kn})$ of type d_5 has a step space which is isomorphic to a subspace of E.*

Proof: Suppose $\sup_n \alpha_{n+1}/\alpha_n = A < +\infty$ and (b_{kn}) satisfies

$$b_{k+1,n}/b_{kn} \leqq (b_{k+2,n}/b_{k+1,n})^M \quad \forall\ k,\ n$$

and for some $M>1$. Construct (p_k) which strictly increases to infinity by $p_1=1$ and

$$\frac{p_{k+1}^{-1} - p_{k+2}^{-1}}{p_k^{-1} - p_{k+1}^{-1}} \leqq (AM)^{-1}.$$

Claim: For any m_0, q_0 there exist $m > m_0$, $q_0 < q_{1m} < \dots < q_{mm}$ satisfying:

$$\text{(2)} \qquad \exp((p_k^{-1} - p_{k+1}^{-1})\,\alpha_{q_{kn}}) \leqq b_{k+1,n}/b_{kn} < \exp((p_k^{-1} - p_{k+1}^{-1})\,\alpha_{q_{k+1,n}})$$

for $k=1, 2, \dots, m$.

Proof of claim: Let m be the smallest index with $m > m_0$ and $\exp((p_1^{-1} - p_2^{-1})\alpha_{q_0+1}) \leqq b_{2m}/b_{1m}$. We choose (q_{kn}) inductively by $q_{1m} = q_0 + 1$ and supposing $q_{1m} < q_{2m} < \dots < q_{km}$ are chosen satisfying (2) (q_{km} satisfies only the left inequality in (2)), we let $q_{k+1,m}$ be the smalles index which satisfies the right inequality of (2). Then $q_{kn} < q_{k+1,n}$ and

$$\exp((p_{k+1}^{-1}-p_{k+2}^{-1})\alpha_{q_{k+1,n}})\leqq \exp((AM)^{-1}(p_k^{-1}-p_{k+1}^{-1})A\alpha_{q_{k+1,n}}-1)$$

$$<(b_{k+1,n}/b_{kn})^{1/M}\leqq b_{k+2,n}/b_{k+1,n}$$

so the left inequality in (2) with $k+1$ is also satisfied and the induction step is concluded which proves the claim.
The rest of the proof is completed similar to the proof of Proposition 2.1.

Theorem 2.3. *Let $E=\Lambda_1(\alpha)$. Then the following are equivalent:*
i) *α is weakly stable;*
ii) *There exists a weakly stable space $\Lambda_\infty(\beta)$ which is isomorphic to a subspace of E;*
iii) *There exists a weakly stable space $\Lambda_t(\beta)$ which is isomorphic to a subspace of $\Lambda_t(\alpha)$, $t=1, \infty$;*
iv) *Every $\Lambda_\infty(\beta)$ has a step space which is isomorphic to a subspace of* E;
v) *Every $\Lambda_1(\beta)$ has a step space which is isomorphic to a subspace of E;*
vi) *Every $\Lambda_\infty(\beta)$ has a step space which is isomorphic to a subspace of $\Lambda_\infty(\alpha)$;*
vii) *Every $K(a_{kn})$ of type d_5 has a step space which os isomorphic to a subspace of E;*
viii) *Every $K(a_{kn})$ of type d_3 has a step space which is isomorphic to a subspace of $\Lambda_\infty(\alpha)$.*

Proof: The implications i→iii→vii, viii; vii→iv, v; viii→vi; ii→iv are either trivial or conclusions of the previous propositions. We need to show i→ii and each of iv, v, vi→i. To show i→ii we suppose that $\sup_n \alpha_{n+1}/\alpha_n=A<+\infty$ and let $\beta_n=A^n$. Then $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_1(\beta)$ by Corollary 4.3 pp. 77 in [4] and since β is equivalent to a subsequence of α by Lemma 1.2. this proves ii. To prove the remaining implications iv, v, vi→i, suppose α is not weakly stable. Then there is a subsequence (α_{n_i}) of α with $\lim_i \alpha_{n_i}/\alpha_{n_i+1}=0$. Let $\beta_i=(\alpha_{n_i+1}\ \alpha_{n_i})^{\frac{1}{2}}$. Arrange the sequences α and β into a single sequence in a non-decreasing way:

$$\ldots\leqq\alpha_{n_i}<\beta_i<\alpha_{n_i+1}<\ \ldots\ .\ \text{Then } \lim_i \alpha_{n_i}/\beta_i=\lim_i \beta_i/\alpha_{n_i+1}=0.$$

From [2] this is the exact condition for every linear continuous mapping $T:\Lambda_t(\beta)\to\Lambda_t(\alpha)$ to be compact, $t=1, \infty$. So v, vi cannot hold. Moreover, this condition also implies that every linear continuous mapping $T:\Lambda_\infty(\beta)\to\Lambda_1(\alpha)$ is compact by [3] so iv cannot hold.

Proposition 2.4. *A G_∞ space E is either isomorphic to a weakly stable $\Lambda_\infty(\alpha)$ or contains no subspaces isomorphic to any weakly stable $\Lambda_\infty(\beta)$.*

Proof: Suppose there exists a weakly stable $\Lambda_\infty(\beta)$ which is isomorphic to a subspace of E. Then by a theorem of M. S. Ramanujan and T. Terzioglu [7] E itself is isomorphic to some $\Lambda_\infty(\alpha)$. So the condition iii of Theorem 2.3. is satisfied so α must be weakly stable.

We next consider the problem of embedding $\Lambda_\infty(\alpha)$ into $\Lambda_i(\alpha)$. We start with a technical lemma:

Lemma 2.5. *Let $E=\Lambda_\infty(\beta)$ be isomorphic to a subspace of $F=\Lambda_1(\alpha)$. Let $T: E\to F$ be the embedding given by $Te_n=y_n=\Sigma_i t_{ni} f_i$, where (e_n), (f_n) denote the coordinate bases in E resp. F. Let (s_k), (p_k) be sequences of positive numbers strictly increasing to infinity so that with*

$$a_{kn}=\exp(-\alpha_n/s_k),\quad b_{kn}=|t_{nq_{kn}}|a_{kq_{kn}},\quad c_{kn}=\exp(p_k\beta_n)$$

the inequality $c_{kn}\leqq b_{kn}\leqq c_{k+1,n}$ is satisfied for each k and for large n. Without loss of generality assume $\alpha_n=\gamma_p$ for $r_p<n\leqq r_{p+1}$, where $\inf_p \gamma_{p+1}/\gamma_p>1$ *and (r_p) is a subsequence of natural numbers. Then:*

i) *If for each k, n we denote by z_{kn} the unique index satisfying $r_{z_{kn}}<q_{kn}\leqq r_{z_{kn}+1}$ then $\forall\ p\ \exists\ n_p$ such that* $\sup_{n\geqq n_p}$ Max $\{k:\ z_{kn}=z_{pn}\}<+\infty$;

ii) $\forall\ s\ \exists\ k_s$ *such that the set $\{n:\ q_{k_s n}-n\leqq s\}$ is at most finite.*

Proof: i) Suppose the conclusion does not hold. Then there is a p for which we can construct a subsequence (n_j) of N with Max $\{k{:}z_{kn_j}=z_{pn_j}\}\geqq j$ for all j. Now since q_{kn} and hence z_{kn} are non-decreasing in k for a fixed n, if follows that $z_{pn_j}=z_{p+1,n_j}=\ \dots\ =z_{jn_j}$ and hence for any k, k' with $p\leqq k\leqq j$ and $p\leqq k'\leqq j$ we have $\alpha_{q_{knj}}=\alpha_{q_{k'nj}}=\alpha_{q_{pnj}}$ so we obtain:

$$|t_{n_j q_{kn_j}}|=b_{kn_j}/a_{kq_{kn_j}}\geqq|t_{n_j q_{k'n_j}}|a_{kq_{k'n_j}}/a_{kq_{kn_j}}=|t_{n_j q_{k'n_j}}|$$

Therefore by symmetry $|t_{n_j q_{knj}}|=|t_{n_j q_{k'nj}}|$ and hence $b_{kn_j}=|t_{n_j q_{pnj}}|a_{kq_{pnj}}\forall\, j\geqq k(\geqq p)$. Passing to a subsequence of (n_j) if necessary, we can assume $q_{pn_j}\neq q_{pn_{j'}}$ for $j\neq j'$, which means that $\Lambda_\infty(\beta_{n_j})$ is isomorphic to $\Lambda_1(\alpha_{q_{pn_j}})$ which means that $\Lambda_\infty(\beta_{n_j})$ is isomorphic to $\Lambda_1(\alpha_{q_{pn_j}})$ which is a contradiction since any linear continuous mapping T: $\Lambda_1(a)\to\Lambda_\infty(b)$ is compact for any a, b (cf. [8]).

ii) Assume the conclusion does not hold. Then each $M_k=\{n:\ q_{kn}-n\ \leqq s\}$ is infinite. Moreover, since $q_{kn}\leqq q_{k+1,n}$ A n, k we have $M_k\supset M_{k+1}$. Choose a diagonal sequence (n_j) with

$$q_{kn_j}-n_j\leqq s\qquad\forall\ j\geqq k \tag{3}$$

Then by the fundamental inequality we obtain:

$$\exp[(p_{k+1}-p_k)\beta_n]=c_{k+1,n}/c_{kn}\leqq b_{k+1,n}/b_{k-1,n}$$

$$\leqq a_{k+1,q_{k+1,n}}/a_{k-1,q_{k+1,n}}=\exp[(s_{k-1}^{-1}-s_{k+1}^{-1})\,\alpha_{q_{k+1,n}}]$$

that is, for large n we obtain

$$(4) \qquad \beta_n/\alpha_{q_{kn}} \leqq \frac{s_{k-1}^{-1} - s_{k+1}^{-1}}{p_{k+1} - p_k} = C_k.$$

Fix $k_0 > 1$. By part i with $p = k_0$ we find indices m_{k_0} and k_1 with $z_{k_1+1,n} > z_{k_0 n}$ $\forall\ n \geqq m_{k_1}$. Applying part i with $p = k_1 + 1$ then we find m_{k_1} and k_2 etc. Hence after successive applications we obtain:

$$(5) \qquad \forall\ q \quad \exists\ k_q \text{ and } m_{k_{q-1}} \text{ with } z_{k_q+1,n} \geqq z_{k_0 n} + q \ \forall\ n \geqq m_{k_{q-1}}.$$

By the hypothesis there exists some $B > 0$ such that $\sup_n \alpha_n/\beta_n = B < +\infty$ (cf. [4]). Let s_0 be large enough so that $BC_{k_0} < A^{s_0 - s - 1}$ holds. Let j be large enough so that (5) holds with $n = n_j$ and $q = s_0$, (4) holds for $k = k_0$ and $n = n_j$, (3) holds for $k = k_{s_0} + 1$. Then we obtain:

$$BC_{k_0} < A^{s_0 - s - 1} \leqq \gamma_{z_{k_0 n_j}} + s_0 - s/\gamma_{z_{k_0 n_j}} \leqq \gamma_{z_{k_{s_0}+1,n_j}} - s/\alpha_{q_{k_0 n_j}}$$

$$\leqq B\ \beta_{q_{k_{s_0}+1,n_j}} - s/\alpha_{q_{k_0 n_j}} \leqq B\ \beta_{n_j}/\alpha_{q_{k_0 n_j}} \leqq BC_{k_0}$$

which is a contradiction.

Proposition 2.6. *If $\Lambda_\infty(\alpha)$ isomorphic to a subspace of $\Lambda_1(\alpha)$ then α is weakly stable.*

Proof: We use the notation of Lemma 2.5. By part (ii) of lemma 2.5. we find k_0 so that the set $\{n\colon q_{k_0 n} \leqq n\}$ is finite. Let $s_0 = \text{Max}\ \{n\colon q_{k_0 n} \leqq n\} + 1$. By part i of Lemma 2.5. find k_1 and s_1 such that $z_{k_1 n} > z_{k_0 n}$ holds $\forall\ n \geqq s_1$. Then clearly $q_{k_1 n} > q_{k_0 n}$. On the other hand, by the fundamental inequality we obtain $\sup_n \alpha_{q_{kn}}/\alpha_n = Q_k < +\infty$ for any k. Let $m_1 = \text{Max}\ \{s_0, s_1, p_{k_1}\}$ and define (m_j) inductively by $m_j = q_{k_1 m_{j-1}}$. Then $m_{j-1} \leqq q_{k_0 m_{j-1}} < q_{k_1 m_{j-1}} = m_j$ and with $\varepsilon_j = \alpha_{m_j}$ we have $\sup_j \varepsilon_{j+1}/\varepsilon_j = \sup_j \alpha_{q_{k_1 m_j}}/\alpha_{m_j} = Q_{k_1} < +\infty$. Having a weakly stable subsequence, by Lemma 1.5. the sequence α is itself weakly stable.

Proposition 2.7. *If $E = \Lambda_\infty(\alpha)$ is isomorphic to a subspace of $F = \Lambda_1(\alpha)$ spanned by a block basic sequence then* α is stable.

Proof: By Proposition 2.6. α is weakly stable so assume α is given in a standard form with $A > 1$ and (r_p). Let the embedding $T\colon E \to F$ be given by $Te_n = y_n = \Sigma_{i \in N_n} t_{ni} f_i$, where (e_n) and (f_n) are coordinate bases in E resp. F and (N_n) is a sequence of pairwise disjoint, finite subsets of $\mathbb{N}$.

We use the notation of Lemma 2.5. By the fundamental inequality we obtain $\sup_n \alpha_{q_{kn}}/\alpha_n = B_k < +\infty$ for each k, hence

$$A^{z_{kn}}/A^p = \alpha_{q_{kn}}/\alpha_n \leqq B_k \leqq A^{\nu_k}$$

for some ν_k hence $z_{kn} \leqq p + \nu_k$ if $r_p < n \leqq r_{p+1}$. Again invoking the fundamental inequality we obtain for $m \leqq n$:

$$\alpha_{q_{kn}}/\alpha_{q_{k+2,m}} \leqq D_k,$$

hence $A^{z_{kn}}/A^{z_{k+2,m}} = \alpha_{q_{kn}}/\alpha_{q_{k+2,m}} \leqq D_k \leqq A^{\beta_k}$ for some β_k. It follows that

(6) $$z_{kn} \leqq \beta_k + z_{k+2,m} \quad m \geqq n.$$

By part i of Lemma 2.5, $\forall\ p\ \exists\ n_p,\ M_p$ such that Max $\{k\colon z_{kn} = z_{pn}\} = M_p < +\infty$ $\forall\ n \geqq n_p$. Hence inductively we can find k_p, n'_p such that

(7) $$z_{k_1 n} < z_{k_2 n} < \ . \ . \ . \ < z_{k_p n} \quad \forall\ n \geqq n'_p.$$

By part ii) of Lemma 2.5. we find some k_0 and n'' with $q_{k_0 n} \geqq n\ \forall\ n \geqq n''$. Find k_4 and n'_4 by (7). Fix $k = \text{Max}\ \{k_0, k_4\}$ $n_0 = \text{Max}\ \{n'', n_k, 4n'_4\}$. Then for $n \geqq n_0$ we have $q_{kn} \geqq n$,

(8) $$z_{k_1 n} < z_{k_2 n} < z_{k_3 n} < z_{k_4 n} \leqq z_{kn}.$$

Let $P = \{n\colon\ q_{kn} - n \leqq r_p\}$, S_n = cardinality of $\{m\colon\ m \leqq n$ and $q_{km} \leqq q_{kn}\}$.

Claim: $\exists\ \mu$ such that $5r_p/4 \leqq r_{p+\mu}$ for $p \geqq p_0$.

We shall prove the claim in the two cases:

Case 1: $n \geqq n_0,\ n \in P$.

Suppose $4(n - S_n) < r_p$. Then $4(S_n - n'_4) > 4n - r_p - 4n'_4 > 2n > 0$ so there are at least $S_n - n'_4$ indices m which satisfy $n \geqq m \geqq n'_4$ and $q_{km} \leqq q_{kn}$, that is, since by (7) $q_{k_1 m} < \ldots < q_{k_4 m} \leqq q_{km} \leqq q_{kn}$ and q_{km}'s are distinct (since they belong to disjoint sets N_m) this means $4(S_n - n') \leqq q_{kn} \leqq n + r_p$. We obtain the contradiction $4n < r_p + 4S_n \leqq r_p + n + r_p + 4n'_4 \leqq 4n$ and hence we conclude that $4(n - S_n) \geqq r_p$. That is, there are at least $[r_p/4]$ indices m which satisfy $m \leqq n$ and $q_{kn} < q_{km}$. Due to the block basic embedding, all of these q_{km}'s are distinct, hence there exists some $m_1 < n$ such that $[r_p/4] + q_{kn} < q_{km_1}$. Then we obtain:

$$5r_p/4 < n + r_p/4 \leqq q_{kn} + r_p/4 < q_{km_1} + 1 \leqq r_{z_{km_1}+1} + 1 \leqq r_{z_{k+2,n}+\beta_k+2}$$
$$\leqq r_{p+\nu_{k+2}+\beta_k+2} = r_{p+\mu_1}$$

where (8) and (6) are used and $\mu_1 = \nu_{k+2} + \beta_k + 2$.

Case 2: $n \geqq n_0$, $n \notin P$.

In this case $q_{kn} - n > r_p$ and we obtain: $2r_p < r_p + q_{kn} - n \leqq r_{z_{kn}+1} \leqq r_{p+\nu_k+1} = r_{p+\mu_2}$ where $\mu_2 = \nu_k + 1$.

Hence combining the two results we obtain $dr_p \leqq r_{p+\mu}$ for $p \geqq p_0$ (where $r_{p_0} \geqq n_0 > r_{p_0-1}$ holds) where $\mu = \max\{\mu_1, \mu_2\}$, $d = 5/4$. The claim is thus proved.

For $p \geqq p_0$ we have: $2r_p \leqq d^4 r_p \leqq d^3 r_{p+\mu} \leqq \ldots \leqq r_{p+4\mu}$. Define $\beta_n = (A^{4\mu})^p$ if $q_p < n \leqq q_{p+1}$ where $q_p = r_{4\mu(p-1)+1}$, $p \geqq p_0$. Then $2q_p \leqq q_{p+1}$ for $p \geqq p_0$ so β is stable by Lemma 1.4.

It remains to show that $\alpha \sim \beta$: With $q_p < n \leqq q_{p+1}$ we obtain: $\alpha_n/\beta_n \leqq \alpha_{q_{p+1}}/\beta_n = 1$ and $\beta_n/\alpha_n \leqq \beta_n/\alpha_{q_p} = A^{4\mu}$. Hence we conclude that α is stable.

Remark 2.8. We note that the converse of Proposition 2.7. also holds (cf. Corollary 2.4.4 pp. 63 [4]). Hence embeddability of $\Lambda_\infty(\alpha)$ into $\Lambda_1(\alpha)$ through a block basic embedding characterizes α as a stable exponent sequence. Concerning the general embedding of $\Lambda_\infty(\alpha)$ into $\Lambda_1(\alpha)$, Proposition 2.6. states that α is necessarily weakly stable. We shall show that the converse is not true. To construct a counter example we shall first derive a necessary condition.

Proposition 2.9. *If $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ then $\alpha \sim \beta$ which satisfies either* $\lim_n \beta_{n+1}/\beta_n > 1$ *or* $\lim_n \beta_{n+1}/\beta_n = 1$.

Proof: We use the notation of Lemma 2.5. Choose l_0 and k_0 satisfying $s_{l_0}^{-1} \leqq 1$ and $p_{l_0+1} - p_{k_0} \leqq -2$. Then $-j, q, m, l$. We have:

$$I = |t_{mq}|\, a_{jq}/b_{k_0 m} = (|t_{mq}|\, a_{lq}/b_{k_0 m})(a_{jq}/a_{eq}) \leqq (b_{lm}/b_{k_0 m})(a_{jq}/a_{lq}) \leqq (c_{l+1,m}/$$

$$c_{k_0 m})(a_{jq}/a_{lq}) = \exp[(p_{l+1} - p_{k_0})\alpha_m + (s_l^{-1} - s_j^{-1})\alpha_q].$$

Since (y_n) is a basic sequence in $\Lambda_1(\alpha)$ there exists a system $(|\cdot|_k)$ of norms on $\Lambda_1(\alpha)$ equivalent to the usual norm $(\|\cdot\|_k)$ so that (y_n) is a basic sequence in each $[\Lambda_1(\alpha), |\cdot|_k]^\wedge$ (cf. [1]). Hence there exist C, k, j with $C^{-1}\|\cdot\|_{k_0} \leqq |\cdot|_k \leqq C\|\cdot\|_j$. Let ν be such that $p_{j+2} - p_{k_0} + (s_{j+1}^{-1} - s_j^{-1})A^\nu \leqq -1$. Suppose $r_{q'} < q \leqq r_{q'+1}$ and $r_{m'} < m \leqq r_{m'+1}$. Then: if $q' \leqq m'$: take $l = l_0$ and we have

$$I \leqq \exp[(p_{l_0+1} - p_{k_0})\alpha_m + (s_{l_0}^{-1} - s_j^{-1})\alpha_q]$$

$$\leqq \exp(-2\alpha_m + \alpha_q) \leqq \exp(-\alpha_m)$$

if $q' > m' + \nu$: take $l = j+1$ then

$$I \leqq \exp[(p_{j+2} - p_{k_0})\alpha_m + (s_{j+1}^{-1} - s_j^{-1})\alpha_q] \leqq \exp(-\alpha_m)$$

since $\alpha_q = A^{q'} > A^{m'+v} = \alpha_m A^v$.

Hence if $r_{m'+1} < q \leqq r_{m'+1+v}$ does not hold then $I \leqq \exp(-\alpha_m)$. Define $z_m = \sum_{n=r_{m'+1}+1}^{r_{m'+1+v}} t_{mn} e_n$ for $r_{m'} < m \leqq r_{m'+1}$. Then

$$\sum_m \frac{|y_m - z_m|_k}{|y_m|_k} \leqq C^2 \sum_m \sup_{\substack{q \leqq r_{m'+1} \text{ or} \\ q > r_{m'+1+v}}} \frac{|t_{mq}| a_{jq}}{\|y_m\|_{k_0}} < +\infty.$$

Hence for some m_0, $(z_m)_{m \geqq m_0}$ is a basic sequence in $(\Lambda_1(\alpha), |\cdot|_k)\hat{}$ (cf. Lemma 3.4. pp. 69 [4]) hence in particular linearly independent. We have

$$\{z_m:\ r_{m'} < m \leqq r_{m'+1}\} \subset \operatorname{sp}\{e_m:\ r_{m'+1} < m \leqq r_{m'+1+v}\},$$

hence for any s,

$$\{z_m:\ r_{m'} < m \leqq r_{m'+s+1}\} \subset \operatorname{sp}\{e_m:\ r_{m'+1} < m \leqq r_{m'+s+v+1}\}.$$

Hence for $m \geqq m_0$ and any s, $r_{m+s+1} - r_m \leqq r_{m+s+v+1} - r_{m+1}$ or $r_{m+1} - r_m \leqq r_{m+s+v+1} - r_{m+s+1}$. Without loss of generality assume the inequality is valid for all m. Recalling that $\alpha_n = A^p$ for $r_p < n \leqq r_{p+1}$, define $\beta_n = A^{vp}$ for $q_p < n \leqq q_{p+1}$ where $q_p = r_{vp+1}$.
Then if $q_p \leqq n \leqq q_{p+1}$, we have $\alpha_n/\beta_n \leqq \alpha_{q_{p+1}}/A^{vp} = A^v$ and $\beta_n/\alpha_n \leqq A^{vp}/\alpha_{q_p} = 1$. Hence $\alpha \sim \beta$ and

$$q_{m+1} - q_m = r_{v(m+1)+1} - r_{vm+1} = \sum_{i=1}^{v} (r_{vm+1+i} - r_{vm+i})$$

$$\leqq \sum_{i=1}^{v} (r_{vm+i+s_i+v+1} - r_{vm+i+s_i+1})$$

for any $s_i \geqq 0$. Choosing in particular $s_i = (t+1)_v - i$ for any fixed $t > 0$ and writing $\Delta_m = q_{m+1} - q_m$, we have

$$\Delta_m \leqq \sum_{i=1}^{v} (r_{v(m+t+2)+1} - r_{v(m+t+1)+1}) = v \cdot \Delta_{m+t+1}.$$

We conclude that for any $n \geqq m$ we have $\Delta_m \leqq v\Delta_n$. It follows that either $\sup_n \Delta_n < +\infty$ or $\lim_n \Delta_n = +\infty$. Suppose $\sup_n \Delta_n = M < +\infty$ *and let* $B = A^v$. Then define $\gamma_n = \beta_n B^{(n-q_p)/\Delta_p}$ for $q_p < n \leqq q_{p+1}$. Then $\gamma_{n+1}/\gamma_n = \beta^{1/\Delta_p}$ for $q_p < n \leqq q_{p+1}$. Hence $\lim \gamma_{n+1}/\gamma_n \geqq B^{1/M} > 1$. On the other hand, $\gamma \sim \beta$ follows by:

$$1 \leqq \gamma_n/\beta_n = B^{(n-q_p)/\Delta_p} \leqq B.$$

Hence $\alpha \sim \beta \sim \gamma$. To complete the proof suppose $\lim_n \Delta_n = +\infty$. Define $\gamma_n = B^p(1 + \frac{n-q_p}{\Delta_p}(B-1))$ for $q_p < n \leqq q_{p+1}$. Then if $q_p < n < n+1 \leqq q_{p+1}$:

$$\gamma_{n+1}/\gamma_n = 1 + \frac{B-1}{\Delta_p + (n-q_p)(B-1)} \leqq \frac{B-1}{\Delta_p} + 1$$

if $n=q_p$:

$$\gamma_{n+1}/\gamma_n=1+\frac{B-1}{\Delta_p}.$$

Hence $1\leqq\gamma_{n+1}/\gamma_n\leqq 1+(B-1)/\Delta_p$, for $q_p\leqq n<q_{p+1}$ so $\lim_n \gamma_{n+1}/\gamma_n=1$. We also have

$$1\leqq\gamma_n/\beta_n=1+\frac{n-q_p}{\Delta_p}(B-1)\leqq A$$

Therefore $\alpha\sim\beta\sim\gamma$.

Example 2.10. In this example we construct a weakly stable exponent sequence α for which $\Lambda_\infty(\alpha)$ is not isomorphic to a subspace of $\Lambda_1(\alpha)$. Define α in a standard form where $A>1$, $r_1=0$ and for $p>1$: $r_{p+1}=m+r_p$ if $p=m(m+1)/2$ for some $m\geqq 1$ and $r_{p+1}=1+r_p$ otherwise. We shall check that the conclusion of Proposition 2.9 cannot hold. Suppose $\alpha\sim\beta$, say $C^{-1}\leqq\alpha_n/\beta_n\leqq C$ $\forall n$.

i) Assume $\underline{\lim}\ \beta_{n+1}/\beta_n>1$: Take $p=m(m+1)/2$ and hence $r_{p+1}=r_p+m$. Find m_0, $M>1$ such that if $m\geqq m_0$, the $\beta_{n+1}/\beta_n\geqq M$ for all $r_p<n\leqq r_{p+1}$, p of above form. Then we obtain the contradiction that for $m\geqq m_0$:

$$M^{m-1}\leqq\beta_{r_p+m}/\beta_{r_p+1}\leqq C^2\ \alpha_{r_p+m}/\alpha_{r_p+1}=C^2.$$

Hence $\alpha\not\sim\beta$.

ii) Assume $\lim \beta_{n+1}/\beta_n=1$: Take $p=m(m+1)/2+1$ and hence $r_{p+m}=m+r_p$. Choose n_0 such that $\beta_{n+1}/\beta_n<A^{1/2}$ for $n\geqq n_0$. Then if $n>r_p\geqq n_0$ where p has the above form, we obtain the contradiction that:

$$A^m=\alpha_{r_{p+m}}/\alpha_{r_p+1}=\alpha_{r_p+m}/\alpha_{r_p+1}\leqq C^2\ \beta_{r_p+m}/\beta_{r_p+1}\leqq C^2\ A^{(m-1)/2}.$$

Hence we concluded $\alpha\not\sim\beta$. Finally it follows by Proposition 2.9 that $\Lambda_\infty(\alpha)$ is not isomorphic to any subspace of $\Lambda_1(\alpha)$.

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ERRATA

Page	Line	Instead of	Please, read
7	1 from bottom	$V_{M,\lambda_0 m} \cap F(\partial U_{1,\lambda_0}, t) =$	$V_{M,\lambda_0 m} \cap F(\partial U_{1,\lambda_0}, t) = \emptyset$
20	10 from top	$BC_{k_0} < A^{s_0-s-1} \leqq \gamma_{z_{k_0 n_j}+s_0-s} / \gamma_{z_{k_0 n_j}}$	$BC_{k_0} < A^{s_0-s-1}$
		$\leqq \gamma_{z_{k_{s_0}+1, n_j}-s} / \alpha_{q_{k_0 n_j}}$	$\leqq \gamma_{z_{k_0 n_j}+s_0-s} / \gamma_{z_{k_0 n_j}}$
		$\leqq B\ \beta_{q_{k_{s_0}+1, n_j}-s} / \alpha_{q_{k_0 n_j}}$	$\leqq \gamma_{z_{k_{s_0}+1, n_j}-s} / \alpha_{q_{k_0 n_j}}$
		$\leqq B\ \beta_{n_j} / \alpha_{q_{k_0 n_j}} \leqq BC_{k_0}$	$\leqq B\ \beta_{q_{k_{s_0}+1, n_j}-s} / \alpha_{q_{k_0 n_j}}$
			$\leqq B\ \beta_{n_j} / \alpha_{q_{k_0 n_j}} \leqq BC_{k_0}$