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An Operator Theoretic Approach to Analytic Functions into a Grassmann Manifold

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Introduction

Let D be an open and connected subset of $\mathbf{C}^m$, and let $\mathscr{H}$ be a complex, separable, finite or infinite dimensional Hilbert space. For any positive integer n, let Gr. $(n,\mathscr{H})$ denote the Grassmann manifold associated with n and $\mathscr{H}$ i. e., th e set of all n-dimensional subspaces of $\mathscr{H}$. In the main part of this paper we shall consider the class $A_n(D)$ of all Gr $(n,\mathscr{H})$-valued analytic functions defined on D.

Two functions f and $\tilde{f}$ in $A_n(D)$ are called congruent, if there exists a unitary operator on $\mathscr{H}$ which moves each subspace $f(z)$, onto $\tilde{f}(z)$, for all points z in D.

We follow Griffiths (cf., [6]) in saying that the functions f and $\tilde{f}$ have order of contact k, where k is a positive integer, if they agree in an osculation sense up to order k. Two congruent functions have order of contact k for any k.

A more or less expected converse of this remark is contained in the congruence theorem (cf., [2, 8]), which asserts that, under a non-degeneracy condition, two functions in $A_n(D)$ are congruent, if and only if they have order of contact n.

This theorem originates from the already mentioned work of P. Griffiths [6]. In the stated above form, the congruence theorem was proved, in the case $m=1$, by M. Cowen and R. Douglas [2].

The general case was discussed in [8]. Although the methods used in [8] are essentially different from that of [2], just as in [2] the proof given in [8] has the inherent defect to be an indirect one. More precisely, the congruence theorem was obtained as a consequence of a rather deep understanding of the local equivalence of Hermitian holomorphic vector bundles of rank n over D. The trouble with such an approach is that many qualitative simple properties of analytic functions into a Grassmann manifold are inevitably not used explicitly.

The aim of the present paper is to give a new and simpler proof of the congruence theorem. The proof uses certain operator theoretic techniques developed in [1]. In fact, the main results of the paper, Theorem 2.4 and Theorem 3.7, could be regarded as essentialy strengthened version of Theorem A and Theorem B from [1], respectively.

Significant examples of functions in $A_n(D)$ arise, in the case where $\mathscr{H}$ is infinite dimensional, in connection with the class $B_n(D)$ introduced by M. Cowen and R. Douglas (cf., [2, 3, 4]). The elements of this class are m-tuples of commuting operators on $\mathscr{H}$, and to any m-tuple T in $B_n(D)$ there corresponds in an obvious fashion a function f_T from D into Gr $(n, \mathscr{H})$. Using a result proved by R. Curto and N. Salinas (cf., [5, Theorem 2.2]) one obtains that f_T is an analytic function. Moreover, two m-tuples from $B_n(D)$ are simultaneously unitarily equivalent if and only if their associated functions are congruent.

This last remark constitutes a good reason for the study of congruent functions in the class $A_n(D)$.

We now give a brief outline of this paper. Section 1 contains some preliminaries on smooth and analytic functions from D into Gr $(n, \mathscr{H})$. In Section 2 we associate to any function f in $A_n(D)$ and any set $\mathscr{X}$ of bounded linear operators on $\mathscr{H}$, a chain of fields of finite dimensional C^*-algebras over D. The local structure of such an object is presented in Theorem 2.4. The discussion of congruent functions in $A_n(D)$ is carried out in Section 3. The main result of this section, Theorem 3.7, is a consequence of Theorem 2.4, and the congruence theorem appears as a particular case. Finally, in Section 4 we digress in order to relate the results of Section 3 to the Cowen-Douglas class $B_n(D)$.

1. Analytic functions into a Grassmann manifold

Throughout the paper D will denote an open and connected subset of $\mathbf{C}^m$ and $\mathscr{H}$ will be a complex separable, finite or infinite dimensional Hilbert space. Given a positive integer n, we shall denote by Gr $(n, \mathscr{H})$ the set of all n-dimensional subspaces of $\mathscr{H}$.

1.1. If $\mathscr{K}$ is a subset of $\mathscr{H}$, let span $\mathscr{K}$ denote the closed subspace of $\mathscr{H}$ generated by $\mathscr{K}$.

Assume that f is a function from D into Gr $(n, \mathscr{H})$ and let D_0 be an open subset of D. A collection $\{h_\alpha : 1 \leqq \alpha \leqq n\}$ of $\mathscr{H}$-valued functions on D_0 will be referred to as a frame for f over D_0 if (1.1.1) $f(z) = \text{span}\ \{h_\alpha(z) : 1 \leqq \alpha \leqq n;\ z \in D_0\}$.

The frame is called smooth, respectively analytic, if all functions h_α, $1 \leqq \alpha \leqq n$, are smooth, respectively analytic, on D_0.

Definition. *A function $f: D \to$ Gr $(n, \mathscr{H})$ is said to be smooth, respectively analytic, if for any z_0 in D there exist an open neighborhood D_0 of z_0 and a smooth, respectively an analytic, frame for f over D_0. The set of all analytic functions from D into Gr $(n, \mathscr{H})$ will be denoted by $A_n(D)$.*

1.2 Let $\mathscr{L}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on $\mathscr{H}$ and let $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$ be the space of all smooth functions from D into $\mathscr{L}(\mathscr{H})$. With pointwise sum, product and involution, the space $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$ becomes a unital involutive algebra. Identifying each operator in $\mathscr{L}(\mathscr{H})$ with a constant function on D, one obtains a natural inclusion of $\mathscr{L}(\mathscr{H})$ into $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$. The unit of $\mathscr{L}(\mathscr{H})$ will be denoted by 1.

For $\mathscr{K}$ a closed subspace of $\mathscr{H}$, let $[\mathscr{K}]$ denote the selfadjoint projection in $\mathscr{L}(\ \ )$ with the range $\mathscr{K}$. Given a Gr $(n, \mathscr{H})$-valued function f on D we denote by $[f]$ the function defined as follow: $[f]: D \to \mathscr{L}(\mathscr{H})$, $[f](z) = [f(z)]$; $z \in D$. It is clear that f is smooth if and only if $[f]$ is a self-adjoint projection in $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$.

1.3. In order to state the next result we introduce the notations

$$\text{(1.3.1)} \qquad \partial_i = \partial/\partial z_i, \quad \bar{\partial}_i = \partial/\partial \bar{z}_i: \quad 1 \leqq i \leqq m.$$

Proposition. *Let $f: D \to$ Gr $(n, \mathscr{H})$ be a smooth function and let us put $p = [f]$. The following conditions are equivalent:*

(i) f *is analytic;*
(ii) $(1-p)\ \bar\partial_i p=0$; $\quad 1\leqq i\leqq m$.

Proof. Assume that f is analytic and let z_0 be a point in D. Let $\{h_\alpha : 1\leqq\alpha\leqq n\}$ be an analytic frame for f over an open neighborhood D_0 of z_0. From (1.1.1) one obtains that there exists a smooth frame $\{g_\alpha : 1\leqq\alpha\leqq n\}$ for f over D_0 such that

$$p(z)\,h=\sum_{\alpha=1}^{n}\langle h,\ h_\alpha(z)\rangle\ g_\alpha(z);\ z\in D_0,\ h\in\mathcal{H}., \tag{1.3.2}$$

where $\langle\cdot\rangle$ denotes the inner product on $\mathcal{H}$. In fact the functions g_α, $1\leqq\alpha\leqq n$, are real-analytic, hence p is real-analytic, too. From (1.3.2) we have

$$(\partial_i p)\ (z)h=\sum_{\alpha=1}^{n}\langle h,\ h_\alpha(z)\rangle\ (\partial_i\ g_\alpha)\ (z);\quad 1\leqq i\leqq m,$$

hence

$$(\partial_i p)\ (1-p)=0;\quad 1\leqq i\leqq m. \tag{1.3.3}$$

Since $(\partial_i p)^*=\bar\partial_i p$, the conditions (ii) and (1.3.3) are equivalent.

Assume now f is smooth and p satisfies the condition (ii). Let z_0 be a point in D and let $\{h_\alpha : 1\leqq\alpha\leqq n\}$ be a smooth frame for f over an open neighborhood D_0 of z_0. Then we have

$$\bar\partial_i h_\alpha=\bar\partial_i(p\ h_\alpha)=(\bar\partial_i p)\,h_\alpha+p\ \bar\partial_i h_\alpha;\quad 1\leqq i\leqq m. \tag{1.3.4}$$

Since $\bar\partial_i p_\alpha=p\bar\partial_i p$, we obtain that $\bar\partial_i h_\alpha(z)$ belongs to $f(z)$ for all z in D_0. It follows that

$$\bar\partial_i h_\alpha=\sum_{\beta=1}^{n}\xi^i_{\alpha\beta}\,h_\beta;\quad 1\leqq i\leqq,\quad 1\leqq\alpha\leqq n, \tag{1.3.5}$$

where $\{\xi^i_{\alpha\beta} : 1\leqq i\leqq m,\ \ 1\leqq\alpha,\ \beta\leqq n\}$ is a collection of complex-valued smooth functions on D_0.

Using (1.3.5), we find

$$\bar\partial_j\ \bar\partial_i h_\alpha=\sum_{\beta=1}^{n}(\bar\partial_j\ \xi^i{}_{\alpha\beta}+\sum_{\gamma=1}^{n}\xi^i{}_{\alpha\gamma}\ \xi^j{}_{\gamma\beta})h_\beta;\quad 1\leqq i,j\leqq m,\quad 1\leqq\alpha\leqq n. \tag{1.3.6}$$

Let us define the $n\times n$ matrix $\xi=(\xi_{\alpha\beta})$ of (0,1)-forms on D_0, as follows $\xi_{\alpha\beta}=\Sigma_{i=1}^m\ \xi^i_{\alpha\beta}\ d\bar z_i$; $1\leqq\alpha,\ \beta\leqq n$. The exterior derivative d acting on smooth forms can be decomposed to obtain $d=\partial+\bar\partial$; $\partial=\Sigma_{i=1}^m\ \partial_i dz_i$, $\bar\partial=\Sigma_{i=1}^m\ \bar\partial_i d\bar z_i$. Since $\bar\partial_j\bar\partial_i=\bar\partial_i\bar\partial_j$, a simple computation shows that, using a matrix notation, from (1.3.6) we have $\bar\partial\xi-\xi\Lambda\xi=0$

Now, by the well-known generalization of Grothendiek's Theorem proved by B. Malgrange (cf., [7]), it folows that, eventually decreasing D_0, there exists an $n\times n$ matrix $\eta=(\eta_{\alpha\beta})$ of complex-valued smooth functions on D_0, such that

$$\bar\partial\eta+\eta\Lambda\xi=0, \tag{1.3.7}$$

$\eta(z)$ is an invertible matrix: $z\in D_0$. This implies that the collection $\{k_\alpha : 1\leqq\alpha\leqq n\}$ defined by

$$(1.3.8) \qquad k_\alpha : D_0 \to \mathscr{H},\ k_\alpha = \sum_{\beta=1}^{n} \eta_{\alpha\beta}\, h_\beta;\ 1 \leqq \alpha \leqq n,$$

is a smooth frame for f over D_0. From (1.3.8), (1.3.5) and (1.3.7) we have successively

$$\bar{\partial}_i k_\alpha = \sum_{\beta=1}^{n} (\bar{\partial}_i \eta_{\alpha\beta}) h_\beta + \sum_{\gamma=1}^{n} \eta_{\alpha\gamma}\, \bar{\partial}_i h_\gamma = \sum_{\beta=1}^{n} (\bar{\partial}_i \eta_{\alpha\beta} + \sum_{\gamma=1}^{n} \eta_{\alpha\gamma}\, \xi^i{}_{\gamma\beta})\, h_\beta = 0$$

for all $1 \leqq i \leqq m$ and $1 \leqq \alpha \leqq n$. Thus, $\{k_\alpha : 1 \leqq \alpha \leqq n\}$ is a n analytic frame, hence f is analytic.

1.4. Our next task is to give some consequences of Proposition 1.3. Let $(Z^+)^m$ be the set of all *m-tuples* $I = (i_1, \ldots, i_m)$ of nonnegative integers. We shall use the following standard notations: $D_I = (\partial_1)^{i_1} \ldots (\partial_m)^{i_m}$, $\bar{D}_I = (\bar{\partial}_1)^{i_1} \ldots (\bar{\partial}_m)^{i_m}$, $|I| = i_1 + \ldots + i_m$. For any A in $\mathscr{E}\,(D, \mathscr{L}\,(\mathscr{H}))$ we have

$$(1.4.1) \qquad (D_I \bar{D}_J A)^* = D_J \bar{D}_I A^*; \quad I, J \in (Z^+)^m.$$

If $I = (0, \ldots, 0)$, then we put $D_I A = \bar{D}_I A = A$.

Proposition. *Let $p = [f]$ be the self-adjoint projection in $\mathscr{E}\,(D.\,\mathscr{L}\,(\mathscr{H}))$ associated with a function f in the class $A_n(D)$. Then we have*

$$(1.4.2) \qquad (\bar{D}_I p)\, p = 0, \quad \bar{D}_I p = p\, (\bar{D}_I p); \quad |I| \geqq 1,$$

$$(1.4.3) \qquad p\, (D_I p) = 0, \quad D_I p = (D_I p) p; \quad |I| \geqq 1,$$

$$(1.4.4) \qquad D_I \bar{D}_J p = (D_I p)\, (\bar{D}_J p) - (\bar{D}_J p)\, (D_I p); \quad |I| = |J| = 1.$$

P r o o f. For the first relation in (1.4.2) we shall proceed by induction. If $|I| = 1$, then we obtain $\bar{D}_I p = \bar{D}_I(p^2) = (\bar{D}_I p) p + p\, (\bar{D}_I p)$. By proposition 1.3 we have $p\, (\bar{D}_I p) = \bar{D}_I p$, thus $(\bar{D}_I p) p = 0$. For $|I| \geqq 2$ put $I = J + K$ with J, K in $(Z^+)^m$ and $|J| = 1$. Assume that $(\bar{D}_K p) p = 0$. Since $(\bar{D}_J p) p = 0$, it follows

$$0 = (\bar{D}_J\, ((\bar{D}_K p) p)) p = (\bar{D}_I p) p + (\bar{D}_K p)\, (\bar{D}_J p) p = (\bar{D}_I p) p.$$

For the second relation in (1.4.2) we proceed by induction, too. If $|I| = 1$, then we already know that $\bar{D}_I p = p\, \bar{D}_I p$. For $|I| \geqq 2$ we put $I = J + K$ as above and assume that $\bar{D}_K p = p \bar{D}_K p$. Since $(\bar{D}_J p) p = 0$ we have

$$\bar{D}_I p = \bar{D}_J\, (p \bar{D}_K p) = (\bar{D}_J p)\, (\bar{D}_K p) + p \bar{D}_I p = p \bar{D}_I p,$$

and the proof of (1.4.2) is complete.

The relations (1.4.3) are obtained from (1.4.2) using (1.4.1). Finally, if I and J are such that $|I| = |J| = 1$, then using $\bar{D}_J p = p\, (\bar{D}_J p)$ and $p\, (D_I p) = 0$, we have succesively

$$D_I \bar{D}_J p = D_I\, (p \bar{D}_J p) = (D_I p)\, (\bar{D}_J p) + p\, (D_I \bar{D}_J p) = (D_I p)\, (\bar{D}_J p) + \bar{D}_J\, (p D_I p) - (\bar{D}_J p)\, (D_I p)$$
$$= (D_I p)\, (\bar{D}_J p) - (\bar{D}_J p)\, (D_I p).$$

1.5. From (1.4.2) and (1.4.3) we obtain that $(D_I p)\, (D_J p) = 0 = (\bar{D}_J p)\, (\bar{D}_I P)$; $|I|$, $|J| \geqq 1$. Then, by a repeated use of (1.4.4), clearly we have

Lemma. *For any I and J the derivative $D_I\bar{D}_J p$ can be expressed as a sum of monomials of the following two types*

(i) $\pm(D_{I_1} p)(\bar{D}_{J_1} p) \dots (D_{I_k} p)(\bar{D}_{J_k} p)$

(ii) $\pm(\bar{D}_{J_1} p)(D_{I_1} p) \dots (\bar{D}_{J_k} p)(D_{I_k} p)$,

where $k>1$ and $I_1+\dots+I_k=I$, $J_1+\dots+J_k=J$.

1.6. For the rest of this section we assume that f is a function in $A_n(D)$ and $p=[f]$. We know that p is real-analytic. Define

(1.6.1) $$\mathscr{E}(f)=\text{span}\,\{p(z)h;\ z\in D,\ h\in\mathscr{H}\}.$$

This subspace of $\mathscr{H}$ will be referred to as the essential space of f. For any z_0 in D we also introduce

(1.6.2) $$\mathscr{E}(f;\ z_0)=\text{span}\,\{D_I p(z_0)h;\ I\in(Z^+)^m,\ h\in\mathscr{H}\}.$$

Lemma. $\mathscr{E}(f;\ z_0)=\mathscr{E}(f)$ *holds.*

Proof. Consider the projections $E_0=[\mathscr{E}(f;\ z_0)]$ and $E=[\mathscr{E}(f)]$. We clearly have $E_0=E_0E$ and also $E_0D_Ip(z_0)=D_Ip(z_0)\,;I\in(Z^+)^m$. Since $\bar{D}_Jp=p\bar{D}_Jp$ one finds $E_0\bar{D}_Jp(z_0)=\bar{D}_Jp(z_0)$; $J\in(Z^+)^m$, and by Lemma 1.5. we obtain

$$E_0D_I\bar{D}_Jp(z_0)=D_I\bar{D}_Jp(z_0);\quad I,\ J\in(Z^+)^m.$$

Since the function p is real-analytic, we conclude that there exists an open subset D_0 of D such that $E_0p(z)=p(z)$; $z\in D_0$.

Now, the real-analytic function $D\ni z\to(1-E_0)p(z)\in\mathscr{L}(\mathscr{H})$ vanishes on D_0, hence it vanishes identically on D, that is $E_0p(z)=p(z)$; $z\in D$. Thus $E_0E=E$, hence $E_0=E$.

2. The main technical result

Throughout this section let p denote the self-adjoint projection in $\mathscr{E}(D,\mathscr{L}(\mathscr{H}))$ associated with a function f in $A_n(D)$ and $\mathscr{X}$ be a fixed subset of $\mathscr{L}(\mathscr{H})$, containing the identity operator 1.

2.1. For any nonnegative integer k, consider the following two self-adjoint subsets of $\mathscr{E}(D,\mathscr{L}(\mathscr{H}))$:

$$\varphi^k=\{(\bar{D}_Jp)\,Y^*X(D_Ip):0\leqq|I|,\ \ |J|\leqq k;\ \ X,\ Y\in\mathscr{X}\}$$

$$\tau^k=\{(\bar{D}_Jp)\,Y^*X(D_Ip):0\leqq|I|,\ \ J|\leqq k+1;\ \ |I|+|J|\leqq 2k+1;\ \ X,\ Y\in\mathscr{X}\}.$$

Given a point z in D we put $\varphi^k(z)=\{S(z):S\in\varphi^k\}$; $\tau^k(z)=\{T(z):T\in\tau^k\}$; $\varphi^\infty(z)=\bigcup_{k\geqq 0}\varphi^k(z)$, and let $\mathscr{A}^k(z)$; $\mathscr{B}^k(z)$ and $\mathscr{A}^\infty(z)$ denote the C^*-algebras generated in $\mathscr{L}(\mathscr{H})$ by $\varphi^k(z)$, $\tau^k(z)$ and $\varphi^\infty(z)$, respectively. By Proposition 1.4 one observes that all these C^*-algebras are finite dimensional and have the common unit $p(z)$. Finally, for any open subset D_0 of D let us introduce the following involutive subalgebras of $\mathscr{E}(D_0,\mathscr{L}(\mathscr{H}))$:

$$\Gamma(D_0,\mathscr{A}^k)=\{A\in\mathscr{E}(D_0,\mathscr{L}(\mathscr{H})):A(z)\in\mathscr{A}^k(z),\ z\in D_0\},$$

$$\Gamma(D_0, \mathscr{B}^k) = \{A \in \mathscr{E}(D_0, \mathscr{L}(\mathscr{H})) : A(z) \in \mathscr{B}^k(z),\ z \in D_0\},$$

$$\Gamma(D_0, \mathscr{A}^\infty) = \{A \in \mathscr{E}(D_0, \mathscr{L}(\mathscr{H})) : A(z) \in \mathscr{A}^\infty(z),\ z \in D_0\}.$$

Obviously we have $\Gamma(D_0, \mathscr{A}^k) \subset \Gamma(D_0, \mathscr{B}^k) \subset \Gamma(D_0, \mathscr{A}^{k+1}) \subset \Gamma(D_0, \mathscr{A}^\infty)$.

The next two results are direct consequences of Proposition 1.4 and Lemma 1.5. The proofs are simple, therefore we shall omit them.

2.2. Lemma. *For any* A *in* $\Gamma(D_0, \mathscr{A}^k)$ *and* $|I| = |J| = 1$ *we have* $p(D_I A) \in \Gamma(D_0, \mathscr{B}^k)$; $(\bar{D}_J A)p \in \Gamma(D_0, \mathscr{B}^k)$; $p(D_I \bar{D}_J A)p \in \Gamma(D_0, \mathscr{A}^{k+1})$.

2.3. Lemma. *Let* D_0 *and* k *be such that* $p(D_I A)$ *belongs to* $\Gamma(D_0, \mathscr{A}^k)$ *for any* A *in* $\Gamma(D_0, \mathscr{A}^k)$ *and* $|I| = 1$.
Then $\Gamma(D_0, \mathscr{A}^k) = \Gamma(D_0, \mathscr{A}^\infty)$.

Now we are ready to state the main technical result of the paper.

2.4. Theorem. *There exist an open nonempty subset* D_0 *of* D *and an integer* $1 \leqq k \leqq n$, with the properties:

(i) $\Gamma(D_0), \mathscr{A}^k) = \Gamma(D_0, \mathscr{A}^\infty)$,

(ii) *if* $\varphi : \Gamma(D_0, \mathscr{A}^\infty) \to \mathscr{E}(D_0, \mathscr{L}(\mathscr{H}))$ *is a morphism of complex algebras which satisfies*

(2.4.1.) $$\varphi(p(D_I \bar{D}_J A)p) = \varphi(p)(D_I \bar{D}_J \varphi(A))\varphi(p)$$

for all A *in* $\Gamma(D_0, \mathscr{A}^{k-1})$ *and* $0 \leqq |I|, |J| \leqq 1$, *then*

(2.4.1) $$\varphi(p(D_I \bar{D}_J A)p) = \varphi(p) D_I \bar{D}_J \varphi(A)\ \varphi(p)$$

for all A *in* $\Gamma(D_0, \mathscr{A}^\infty)$ *and all* I, J *in* $(Z^+)^m$.

2.5. This theorem is a strengthened version of Theorem A from [1]. At the present moment, using the results of Section 1, its proof is more or less similar with the proof of Theorem A given in [1]. However, for the reader's convenience, we prefer to include in what follows a complete proof.

We begin with a well-known result (see for instance [2], Lemma 3.4 and [9]).

2.6. Lemma. *Let* A *be a self-adjoint element of* $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$ *such that* $A = pAp$. *Then there exist:*

(i) *an open nonempty subset* D_0 of D;

(ii) *a collection* $\{p_\alpha : 1 \leqq \alpha \leqq l\}$ *of self-adjoint orthogonal projection in* $\mathscr{E}(D_0, \mathscr{L}(\mathscr{H}))$;

(iii) *a collection* $\{\mu_\alpha : 1 \leqq \alpha \leqq l\}$ *of real-valued smooth functions on* D_0, *with* $\mu_\alpha(z) \neq \mu_\beta(z)$, $z \in D_0$, $\alpha \neq \beta$, *related as follows:*

$$p(z) = \sum_{\alpha=1}^{l} p_\alpha(z);\ z \in D_0,\ A(z) = \sum_{\alpha=1}^{l} \mu_\alpha(z) p_\alpha(z);\quad z \in D_0.$$

Moreover, from the preceding relations one obtains

$$p_\alpha(z) = \prod_{\beta \neq \alpha} (A(z) - \mu_\beta(z) p_\beta(z)) / (\mu_\alpha(z) - \mu_\beta(z));\ z \in D_0.$$

2.7. Now we return to the finite dimensional C^*-algebras $\mathscr{A}^k(z)$, $\mathscr{B}^k(z)$ and $\mathscr{A}^\infty(z)$ associated with p and $\mathscr{X}$. Given a finite dimensional C^*-algebra $\mathscr{A}$ we denote by $d(\mathscr{A})$ the cardinal of any maximal set of mutually orthogonal self-adjoint minimal projections in $\mathscr{A}$, and let us put

(2.7.1) $$d_z^k = d(\mathscr{A}^k(z));\ d_z^\infty = d(\mathscr{A}^\infty(z)).$$

Of course we have

(2.7.2) $$d_z^0 \leqq d_z^1 \leqq \ldots \leqq d_z^k \leqq \ldots \leqq d_z^\infty \leqq n$$

therefore we can find an open nonempty subset D_0 of D and an integer $1 \leqq k \leqq n$ such that

(2.7.3) $$d_z^{k-1} = d_z^k;\ z \in D_0.$$

Moreover, using some well-known facts about the structure of finite dimensional C^*-algebras (see for instance [10, Chap. I, § 11]), by a repeated use of Lemma 2.6. and eventually decreasing D_0, we may suppose in what follows that there exist:

(i) a sequence $d_1, \ldots, d_l$ of positive integers;

(ii) a system $Q_1, \ldots, Q_l$ of mutually orthogonal self-adjoint central projections in $\Gamma(D_0, \mathscr{A}^{k-1})$;

(iii) a collection $p_\alpha^i : 1 \leqq i \leqq l,\ 1 \leqq \alpha \leqq d_i$, of mutually orthogonal self-adjoint projections in $\Gamma(D_0, \mathscr{A}^{k-1})$;

(iv) a collection $U_{\alpha\beta}^i : 1 \leqq i \leqq l, 1 \leqq \alpha, \beta \leqq d_i$, of elements of $\Gamma(D_0, \mathscr{B}^{k-1})$, such that

(2.7.4) $$d_1 + \ldots + d_l = d_z^{k-1};\ z \in D_0$$

(2.7.5) $$\sum_{\alpha=1}^{d_i} p_\alpha^i = Q_i,$$

$$U_{\alpha\alpha}^i = p_\alpha^i,\ U_{\alpha\beta}^{i*} = U_{\beta\alpha}^i,\ U_{\alpha\beta}^i\ U_{\gamma\delta}^i = \Delta_{\beta\gamma} U_{\alpha\delta}^i,$$

where $\Delta_{\beta\gamma}$ means the Kronecker symbol.

Clearly, by (2.7.5) we obtain that all Q_i, $1 \leqq i \leqq l$, are central projections in $\Gamma(D_0, \mathscr{A}^{k-1})$. By (2.7.4) and (2.7.3) we conclude that all $p_\alpha^i(z)$, $1 \leqq i \leqq l$, $1 \leqq \alpha \leqq d_i$, are minimal projections in $\mathscr{A}_{(z)}^{k-1}$ and also in $\mathscr{A}_{(z)}^k$, for any z in D_0. Now, given I in $(Z^+)^m$ with $|I| = 1$ and $1 \leqq i \leqq l$, we know from Lemma 2.2 that $p(D_I Q_i)$ is in $\Gamma(D_0, \mathscr{B}^{k-1})$ and since Q_i is a central projection in $\Gamma(D_0, \mathscr{B}^{k-1})$ we have $p(D_I Q_i) = pD_I(Q_i Q_i) = p(D_I Q_i)Q_i + Q_i(D_I Q_i) = 2Q_i(D_I Q_i)Q_i$ whence in follows $p(D_I Q_i) = 0 = (D_I Q_i)p$.

From this last relation it is easy to check that $Q_i p(D_I D_J B)p = p(D_I D_J B)pQ_i$; $1 \leqq i \leqq l$, for all B in $\Gamma(D_0, \mathscr{B}^{k-1})$ and $0 \leqq |I| + |J| \leqq 1$. In particular one obtains that Q_i, $1 \leqq i \leqq l$, are central projections in $\Gamma(D_0, \mathscr{A}^k)$.

Now, since the projections $p_\alpha^i(z)$, $1 \leqq i \leqq l$, $1 \leqq \alpha \leqq d_i$, are minimal in $\mathscr{A}_{(z)}^k$ for any z in D_0, by (2.7.6) and the preceding remark we have that, for any A in $\Gamma(D_0, \mathscr{A}^k)$, there exists a uniquely determined collection of complex-valued smooth functions on D_0, $\{\mu_{\alpha\beta}^i(A) : 1 \leqq i \leqq l,\ 1 \leqq \alpha, \beta \leqq d_i\}$ such that

(2.7.6) $$A = \sum_{i=1}^{l} \sum_{\alpha,\beta=1}^{d_i} \mu_{\alpha\beta}^i(A)\ U_{\alpha\beta}^i$$

2.8. Let D_0 and k be as above and let $\varphi : \Gamma(D_0, \mathscr{A}^\infty) \to \mathscr{E}(D_0, \mathscr{L}(\mathscr{H}))$ be a morphism of complex algebras. In order to prove Theorem 2.4, it sufficies to show that

(2.8.1) $$pD_I\, U_{\alpha\beta}^i \in \Gamma(D_0, \mathscr{A}^k),$$

(2.8.2) $$\varphi(p(D_I U^i_{\alpha\beta})p)=\varphi(p)(D_I\varphi(U^i_{\alpha\beta}))\varphi(p),$$

(2.8.3) $$\varphi(p(\bar{D}_I U^i_{\alpha\beta})p)=\varphi(p)(\bar{D}_I\varphi(U^i_{\alpha\beta}))\varphi(p),$$

for all $|I|=1$, $1\leqq i\leqq l$, $1\leqq\alpha,\beta\leqq d_i$.

Indeed, (i) of Theorem 2.4 will be a consequence of (2.8.1), (2.7.6) and Lemma 2.3, and (ii) will follow from (2.8.2), (2.8.3) and (2.7.6).

Our next task is to prove (2.8.1), (2.8.2) and (2.8.3). Consider the subsets of $\Gamma(D_0,\mathscr{B}^{k-1})$ defined by

$$\xi_i=\{p^i_\alpha A(D_K\bar{D}_L B)Cp^i_\beta : 1\leqq\alpha,\beta\leqq d_i,$$

$$A,B,C\in\Gamma(D_0,\mathscr{A}^{k-1}),\ 0\leqq|K|+|L|\leqq 1;\ 1\leqq i\leqq l\}.$$

If z is a point in D_0 then each $U^i_{\alpha\beta}(z)$ is a finite product of elements belonging to ξ_i, evalued in z. Therefore, eventually decreasing D_0, we may suppose that any $U^i_{\alpha\beta}$ is a finite product of $U^i_{\gamma\delta}$'s belonging to ξ_i. Thus we are allowed to prove (2.8.1), (2.8.2) and (2.8.3) assuming that $U^i_{\alpha\beta}$ is an element of ξ_i.

Let us first assume that $U^i_{\alpha\beta}=p^i_\alpha A(D_K B)Cp^i_\beta$ where A,B,C are in $\Gamma(D_0,\mathscr{A}^{k-1})$ and $|K|=1$.

Given I in $(Z^+)^m$ with $|I|=1$, we derive easily that

(2.8.4) $$p(D_I U^{i*}_{\alpha\beta}),\quad (\bar{D}_I U^i_{\alpha\beta})p\in\Gamma(D_0,\mathscr{A}^k),$$

and using (2.4.1), we also find

(2.8.5.) $$\varphi(p(D_I U^{i*}_{\alpha\beta})p)=\varphi(p)(D_I\varphi(U^{i*}_{\alpha\beta}))\varphi(p),$$

(2.8.6) $$\varphi(p(\bar{D}_I U^i_{\alpha\beta})p)=\varphi(p)(\bar{D}_I\varphi(U^i_{\alpha\beta}))\varphi(p).$$

The rest of the proof will be based on the following simple result.

Lemma. Let $\mathscr{A}$ be an involutive algebra and let V, W in $\mathscr{A}$ be given such that $VWV=V$. Then for each derivation δ on $\mathscr{A}$ we have

(2.8.7) $$\delta V=V(\delta F)+\delta(E)V-V(\delta W)V.$$

where $F=WV$ and $E=VW$.

Proof of Lemma. Since $EV=V$ we obtain $V(\delta F)+\delta(E)V=V(\delta W)V+VW(\delta V)+\delta(E)V=V(\delta W)V+E(\delta V)+\delta(E)V=V(\delta W)V+\delta V$. Now let us put in (2.8.7) $\delta=D_I$, $V=U^i_{\alpha\beta}$, $W=U^{i*}_{\alpha\beta}$. Clearly $E=p^i_\alpha$; $F=p^i_\beta$ and we find

(2.8.8) $$D_I U^i_{\alpha\beta}=U^i_{\alpha\beta}(D_I p^i_\beta)+(D_I p^i_\alpha)U^i_{\alpha\beta}-U^i_{\alpha\beta}(D_I U^{i*}_{\alpha\beta})U^i_{\alpha\beta}.$$

Since $p(D_I p^i_\alpha)\in\Gamma(D_0,\mathscr{A}^k)$, from (2.8.8) and (2.8.4) it follows that $p(D_I U^i_{\alpha\beta})\in\Gamma(D_0,\mathscr{A}^k)$. On the other hand, if we put in (2.8.7) $\delta=D_I$ and $V=\varphi(U^i_{\alpha\beta})$, $W=\varphi(U^{i*}_{\alpha\beta})$ then $E=\varphi(p^i_\alpha)$, $F=\varphi(p^i_\beta)$ and we obtain

(2.8.9) $$D_I\varphi(U^i_{\alpha\beta})=\varphi(U^i_{\alpha\beta})(D_I\varphi(p^i_\beta))+\varphi(D_I p^i_\alpha)\varphi(U^i_{\alpha\beta})$$
$$-\varphi(U^i_{\alpha\beta})(D_I\varphi(U^{i*}_{\alpha\beta}))\varphi(U^i_{\alpha\beta}).$$

Using (2.8.5), (2.8.8) and (2.4.1), from (2.8.9) it follows $\varphi(p(D_I U^i_{\alpha\beta})p) = \varphi(p)(D_I \varphi(U^i{}_{\alpha\beta}))\varphi(p)$.

Thus (2.8.1), (2.8.2) and (2.8.3) are proved.

For the second case, when $U^i_{\alpha\beta}=p^i_\alpha A(\bar{D}_L \beta)Cp^i_\beta$, we proceed analogously. The proof of Theorem 2.4 is complete.

3. The congruence theorem

Let f and $\tilde{f}$ be two functions in the class $A_n(D)$. Denote by p and $\tilde{p}$ the self-adjoint projections in $\mathscr{E}(D, \mathscr{L}(\mathscr{H}))$ associated with f and $\tilde{f}$, respectively.

3.1. Definition (cf. [6, 2]). *The functions f and $\tilde{f}$ are called congruent, if there exists a unitary operator U in $\mathscr{L}(\mathscr{H})$ such that $Up(z)=\tilde{p}(z)U$; $z\in D$.*

3.2. Definition (cf., [6, 2]. *Let k be a nonnegative integer. The functions f and $\tilde{f}$ are said to have order of contact k, if for any point z in D there exist:*

(i) *an open neighborhood D_0* of z;

(ii) *two analytic frames $\{h_\alpha : 1\leqq\alpha\leqq n\}$ and $\{\tilde{h}_\alpha : 1\leqq\alpha\leqq n\}$ for f, respectively $\tilde{f}$, over D_0;*

(iii) *a unitary operator U_z in $\mathscr{L}(\mathscr{H})$, such that $U_z D_I h_\alpha(z)=D_I\tilde{h}_\alpha(z)$; $I\in(Z^+)^m$, $0\leqq|I|\leqq k$; $1\leqq\alpha\leqq n$.*

It is not difficult to see that we have:

3.3. Lemma. *The functions f and $\tilde{f}$ have order of contact k, if and only if for any point z in D there exists a unitary operator U_Z in $\mathscr{L}(\mathscr{H})$* such that

$$U_Z D_I p(z)=D_I\tilde{p}(z)\ U_z;\quad I\in(Z^+)^m,\ 0\leqq|I|\leqq k. \tag{3.3.1}$$

As a consequence, if f and $\tilde{f}$ are congruent then they have order of contact k for any k.

3.4. Before continuing we make another remark. Let U_z be as above and consider $V_z=U_z p(z)$. From (3.3.1) one obtains $V_z^* V_z=p(z)$, $V_z V_z^*=\tilde{p}(z)$; hence V_z is a partial isometry in $\mathscr{L}(\mathscr{H})$. Moreover, from (3.3.1) one finds $V_z\bar{D}_J p(z)\, D_I p(z)\ V_z^* =\bar{D}_J\tilde{p}(z)D_I\tilde{p}(z)$; I, $J\in(Z^+)^m$, $0\leqq|I|$, $|J|\leqq k$.

3.5. The congruence theorem. *Let f, $\tilde{f}$ be two functions in $A_n(D)$ such that*

$$\mathscr{E}(f)=\mathscr{E}(\tilde{f})=\mathscr{H} \tag{3.5.1}$$

The following conditions are equivalent:

(i) *f and $\tilde{f}$ are congruent;*

(ii) *f and $\tilde{f}$ have order of contact n;*

(iii) *for any z in D there exists a partial isometry V_z in $\mathscr{L}(\mathscr{H})$ so that $V_z^* V_z=p(z)$, $V_z V_z^*=\tilde{p}(z)$, $V_z\bar{D}_J p(z)D_I p(z)\ V_z^*=\bar{D}_J\tilde{p}(z)D_I\tilde{p}(z)$; $0\leqq|I|$, $|J|\leqq n$.*

3.6. Clearly we have to prove only that (iii) implies (i). This will follow from the next theorem, which is a generalisation of Theorem B from [1]. In order to state it we need some notation. Let f and f be as above and let $\mathscr{X}$ be a subset of $\mathscr{L}(\mathscr{H})$ containing the identity operator 1. Assume that the condition (3.5.1) is satisfied and consider a map $\psi: \mathscr{X}\to\mathscr{L}(\mathscr{H})$ such that $\psi(1)=1$.

3.6. Theorem. *The following conditions are equivalent:*

(i) ψ *is the restriction of a inner automorphism in* $\mathcal{L}(\mathcal{H})$ *induced by a unitary operator* U, *which satisfies* $Up(z)U^*=\tilde{p}(z)$, $z\in D$;

(ii) *for any* z *in* D *there exists a partial isometry* V_z *in* $\mathcal{L}(\mathcal{H})$ *so that*

$$(3.6.1)\qquad V_z^*V_z=p(z);\quad V_zV_z^*=\tilde{p}(z),$$

$$(3.6.2)\qquad V_z\bar{D}_Jp(z)Y^*XD_Ip(z)V_z^*=\bar{D}_J\tilde{p}(z)\psi(Y)^*\psi(X)D_I\tilde{p}(z),$$

for all X, Y, *in* $\mathcal{X}$ *and* $0\leq|I|$, $|J|\leq n$.

Proof. It is clear that (i) implies (ii). The converse is based on Theorem 2.4. We associate with f and $\mathcal{X}$ the open nonempty subset D_0 of D and the integer $1\leq k\leq n$ which appear in Theorem 2.4. Given A in $\Gamma(D_0,\mathcal{A}^\infty)$, let us define $\varphi(A)(z)=V_zA(z)V_z^*$; $z\in D_0$. Since $\Gamma(D_0,\mathcal{A}^\infty)=\Gamma(D_0,\mathcal{A}^k)$, from (3.6.1) and (3.6.2) we obtain that φ is a well-defined morphism of complex algebras from $\Gamma(D_0,\mathcal{A}^\infty)$ into $\mathcal{E}(D_0,\mathcal{L}(\mathcal{H}))$, and the conditions (2.4.1) in Theorem 2.4 are satisfied. Thus, from Theorem 2.4 we conclude by induction that

$$(3.6.3)\qquad V_z\bar{D}_Jp(z)Y^*XD_Ip(z)V_z^*=\bar{D}_J\tilde{p}(z)\psi(Y)^*\psi(X)D_I\tilde{p}(z),$$

for all z in D_0, X and Y in $\mathcal{X}$, I and J in $(Z^+)^m$.

Let z_0 be a fixed point in D_0. Since $I\in\mathcal{X}$, Lemma 1.6 and the assumption (3.5.1) give

$$(3.6.4)\qquad \mathcal{H}=\text{span}\,\{XD_Ip(z_0)h: X\in\mathcal{X},\ I\in(Z^+)^m,\ h\in\mathcal{H}\}=\text{span}\,\{\psi(X)D_I\tilde{p}(z_0)h;\ X\in\mathcal{X},\ I\in(Z^+)^m,\ h\in\mathcal{H}\}.$$

Let U in $\mathcal{L}(\mathcal{H})$ be defined by the equations $U(XD_Ip(z_0)h)=\psi(X)D_Ip(z_0)V_{z_0}h$; $X\in\mathcal{X}$, $I\in(Z^+)^m$, $h\in\mathcal{H}$.

From (3.6.3) and (3.6.4) we derive that U is a unitary operator on $\mathcal{H}$ and also $\psi(X)=UXU^*$; $X\in\mathcal{X}$.

$$(3.6.5)\qquad D_I\tilde{p}(z_0)=UD_Ip(z_0)U^*;\quad I\in(Z^+)^m.$$

Since p and $\tilde{p}$ are real-analytic, using (3.6.5) and the remarks given at 1.5 we conclude that $\tilde{p}(z)=Up(z)U^*$; $z\in D$. The proof is complete.

4. The Cowen-Douglas class $B_n(D)$

Denote as above by D an open and connected subset of $\mathbf{C}^m$ and by $\mathcal{H}$ a separable, infinite dimensional, complex Hilbert space. Given m-tuple $T=(T_1,\ldots,T_m)$ of commuting bounded linear operators on $\mathcal{H}$, and a point $z=(z_1,\ldots,z_m)$ in D, we define

$$\mathcal{K}(T;\ z)=\{h\in\mathcal{H}:\ (z_1-T_1)h=\ \ldots\ =(z_m-T_m)h=0\}.$$

4.1. Definition (cf., [2, 3]). *The m-tuple* T *is said to be in the class* $B_n(D)$, *where* n *is a positive integer, if and only if*

(i) $\dim\mathcal{K}(T;\ z)=n$; $z\in D$,

(ii) span $\bigcup_{z \in D} \mathcal{K}(T;\ z) = \mathcal{H}$.
(iii) range $(z_i - T_i) = \mathcal{H}$, $1 \leqq i \leqq m$, $z \in D$.

4.2. Let $T = (T_1, \ldots, T_m)$ be in $B_n(D)$ and denote by f_T the function defined as follows: $f_T: D \to \mathrm{Gr}(n, \mathcal{H})$, $f_T(z) = \mathcal{K}(T;\ z)$. Arguing as in [3] or using a result of R. Curto and N. Salinas (cf., [4], Theorem 2.2]), one obtains that f_T is analytic. Moreover, we have as a direct consequence of the definitions:

Lemma. *Let* $T = (T_1, \ldots, T_m)$ *and* $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_m)$ *be two m-tuples in the class* $B_n(D)$. *The following conditions are equivalent:*
(i) *there exists a unitary operator* U *on* $\mathcal{H}$ *such that* $UT_i = \tilde{T}_i U$; $1 \leqq i \leqq m$;
(ii) *the functions* f_T *and* $f_{\tilde{T}}$ *are congruent.*

4.3. We now wish to obtain an operator theoretic interpretation of the order of contact of the functions f_T and $f_{\tilde{T}}$. Before stating precisely what we are able to find out, we shall give some preliminary results.

Let $T = (T_1, \ldots, T_m)$ be a fixed m- tuples in $B_n(D)$ and let $p = [f_T]$. For any $z = (z_1, \ldots, z_m)$ in D we have

(4.3.1) $$(z_j - T_j)p(z) = 0; \quad 1 \leqq j \leqq m.$$

Let $I = (i_1, \ldots i_m)$ be in $(Z^+)^m$ and fix $1 \leqq j \leqq m$. If $i_j \geqq 1$, then, differentiating the equation (4.3.1), one obtains

(4.3.2) $$(z_j - T_j)D_I p(z) = -i_j\ D_I(j)p(z),\quad I(j) = (i_1, \ldots, i_j - 1, \ldots, i_m).$$

If $i_j = 0$, then one finds

(4.3.3) $$(z_j - T_j)D_I p(z) = 0$$

Given $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$ in $(Z^+)^m$, put $I - J = (i_1 - j_1, \ldots, i_m - j_m)$. If $I - J$ belongs to $(Z^+)^m$ we write $I \geqq J$. We shall use also the notation $I! = i_1! \ldots i_m!$.

Now, for $z = (z_1, \ldots, z_m)$ in D, let $T^J(z)$ denote the operator in $\mathcal{L}(\mathcal{H})$ defined by

(4.3.4) $$T^J(z) = (z_1 - T_1)^{j_1} \ldots (z_m - T_m)^{j_m}.$$

By a repeated use of (4.3.2) and (4.3.3) we have:

(4.3.5) $$T^J(z)D_I p(z) = (-1)^{|J|}\ (I!/(I-J)!)D_{I-J}\,p(z); \quad I - J \in (Z^+)^m,$$

(4.3.6) $$T^J(z)\ D_I p(z) = 0; \quad I - J \notin (Z^+)^m.$$

Let z be a fixed point in D, and $\{h_\alpha : 1 \leqq \alpha \leqq n\}$ be an analytic frame for f_T over an open neighborhood D_0 of z. For any nonnegative integer k, let us introduce

(4.3.7) $$\mathcal{E}^{(k)}(T;\ z) = \mathrm{span}\ \{D_I h_\alpha(z) : 0 \leqq |I| \leqq k,\ \ 1 \leqq \alpha \leqq n\}.$$

We easily derive

(4.3.8) $$\mathcal{E}^{(k)}(T;\ z) = \mathrm{span}\ \{D_I p(z)h : 0 \leqq |I| \leqq k,\ \ h \in \mathcal{H}\}.$$

From (4.3.5) and (4.3.6) one sees that

(4.3.9) $T^J(z)D_I h_\alpha(z) = \pm(I!/(I-J!)D_{I-J}h_\alpha(z);\ 1 \leqq \alpha \leqq n,\ I-J \in (Z^+)^m$,

(4.3.10) $T^J(z)D_I h_\alpha(z) = 0;\ 1 \leqq \alpha \leqq n,\ I-J \notin (Z^+)^m$.

These equations, together with span $\cup_{k\geqq 0}\mathscr{E}^{(k)}(T;\ z) = \mathscr{E}(f_T;\ z) = \mathscr{H}$ imply that:

Lemma. (i) *The vectors* $\{D_I h_\alpha(z) : I \in (Z^+)^m,\ 1 \leqq \alpha \leqq n\}$ *are independent in* $\mathscr{H}$, *and* span $\{D_I h_\alpha(z) : I \in (Z^+)^m,\ 1 \leqq \alpha \leqq n\} = \mathscr{H}$.

(ii) dim $\mathscr{E}^{(k)}(T;\ z) = (1+m+\ \ldots\ m^k)n$.

4.4. Now we introduce another collection of subspaces: $\mathscr{K}^{(k)}(T;\ z) = \{h \in \mathscr{H}: T^I(z)h = 0,\ |I| = k+1\}$. Obviously $\mathscr{K}^{(0)}(T;\ z) = \mathscr{K}(T;\ z) = \mathscr{E}^{(0)}(T;\ z)$.

Lemma. *For any nonnegative integer* k *we have* $\mathscr{K}^{(k)}(T;\ z) = \mathscr{E}^{(k)}(T;\ z)$.

Proof. By (4.3.9) and (4.3.10) one finds that $\mathscr{E}^{(k)}(T;z) \subset \mathscr{K}^{(k)}(T;\ z)$. Let h be in $\mathscr{K}^{(k)}(T;\ z)$. By Lemma 4.3, there exists a unique collection of complex numbers $\{c_{I,\alpha} : I \in (Z^+)^m,\ 1 \leqq \alpha \leqq n\}$ such that $h = \Sigma_{I,\alpha}\ c_{I,\alpha}\ D_I h_\alpha(z)$.

Let J in $(Z^+)^m$ with $|J| = k+1$. Since $T^J(z)h = 0$, one obtains

$$0 = \sum_{I \geqq J} \sum_{\alpha} \pm C_{I,\alpha}(I!/(I-J)!)D_{I-J}h_\alpha(z).$$

By Lemma 4.3 again, we have $c_{I,\alpha} = 0;\ I \geqq J,\ 1 \leqq \alpha \leqq$ n. Since J is an arbitrary element in $(Z^+)^m$ with $|J| = k+1$, we conclude $c_{I,\alpha} = 0;\ |I| \geqq k+1,\ 1 \leqq \alpha \leqq n$, hence h belongs to $\mathscr{E}^{(k)}(T;\ z)$.

4.5. Let $T = (T_1, \ldots, T_m)$ and $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_m)$ be two m-tuples in $B_n(D)$. Our next task is to give an alternate means of the order of contact. Explicitly, we have:

Proposition. *The following conditions are equivalent:*

(i) f_T *and* $f_{\tilde{T}}$ *have order of contact* k;

(ii) *for any* z *in* D *there exists a unitary operator* $U_z : \mathscr{K}^{(k)}(T;\ z) \to \mathscr{K}^{(k)}(\tilde{T};\ z)$ *so that*

(4.5.1) $\tilde{T}_i | \mathscr{K}^{(k)}(\tilde{T};\ z) = U_z T_i U_z^* | \mathscr{K}^{(k)}(\tilde{T};\ z);\ 1 \leqq i \leqq m$.

Proof. Let $p = [f_T]$ and $\tilde{p} = [f_{\tilde{T}}]$. Assume that f_T and $f_{\tilde{T}}$ have order of contact k and let z be a fixed point in D. Then there exist two analytic frames $\{h_\alpha : 1 \leqq \alpha \leqq n\}$ and $\{\tilde{h}_\alpha : 1 \leqq \alpha \leqq n\}$ for f_T, respectively $f_{\tilde{T}}$, over an open neighborhood D_0 of z, and a unitary operator U_z in $\mathscr{L}(\mathscr{H})$ so that:

(4.5.2) $U_z D_I h_\alpha(z) = D_I \tilde{h}_\alpha(z);\ 0 \leqq |I| \leqq k,\ 1 \leqq \alpha \leqq n$.

By Lemma 4.4 and using the equations (4.3.9), (4.3.10), it follows easily that U_z has the required properties. In order to prove the converse, let us assume that z is a fixed point in D, and let U_z be a unitary operator from $\mathscr{K}^{(k)}(T;\ z)$ onto $\mathscr{K}^{k)}(\tilde{T};\ z)$, satisfying the condition (4.5.1). It is enough to show that

(4.5.3) $D_I \tilde{p}(z) = U_z D_I p(z)\ U_z^*;\ 0 \leqq |I| \leqq k$.

We shall proceed by induction. First we remark that $0=(z_j-\tilde{T}_j)\tilde{p}(z) = U_z(z_j-T_j)U_z^*\tilde{p}(z)$, hence $(z_j-T_j)\ U_z^*\tilde{p}(z)\ \ U_z=0,\ 1\leqq j\leqq n$.

Since dim range $U_z^*\tilde{p}(z)\,U_z=n=$ dim range $p(z)$ we conclude that $U_z^*\tilde{p}(z)\ U_z=p(z)$. Thus (4.5.3) is proved for $|I|=0$.

Assume now that (4.5.3) holds for all $|I|\leqq l$, and let $I=(i_1,\ldots,i_m)$ be such that $|I|=l+1\leqq k$. Let $1\leqq j\leqq m$ with $i_j\geqq 1$. By (4.3.2) one finds $U_z(z_j-T_j)\ U_z^*D_I\tilde{p}(z)=(z_j-\tilde{T}_j)\ D_I\tilde{p}(z)=-i_jD_{I(j)}\tilde{p}(z)$. From the induction assumption one obtains $(z_j-T_j)\ \ U_z^*D_I\tilde{p}(z)\,U_z=-i_j D_{I(j)}\,p(z)$ whence

$$(z_j-T_j)\ (U_z^*D_I\tilde{p}\ (z)\,U_z-D_Ip(z))=0. \tag{4.5.4}$$

By (4.3.3) the equation (4.5.4) is also true if $i_j=0$. Therefore, $U_zD_Ip(z)U_z-D_Ip(z)=p(z)\ \ (U_zD_Ip(z)U_z-D_Ip(z))$. But $p(z)U_z=U_zp(z)$ and $p(z)D_Ip(z)=0=p(z)D_Ip(z)$ (cf., Proposition 1.4), hence $U_zD_Ip(z)U_z=p(z)$.

The proof is complete.

4.6. We are now ready to state the main result of this section. It could be regarded as a generalisation of Theorem 1.6 from [2] (see also [1], Theorem C).

Let $T=(T_1,\ldots,T_m)$ and $\tilde{T}=(\tilde{T}_1,\ldots,\tilde{T}_m)$ be two m-tuples in $B_n\,(D)$. We denote by T' the commutant of $\{T_1,\ldots,T_m\}$ and assume that $\mathscr{X}$ is a subset of T' containing the identity operator 1 and all operators $T_1,\ldots,T_m$.

Let $\psi:\mathscr{X}\to\mathscr{L}\ (\mathscr{H})$ be a map such that

$$\psi(1)=1,\quad \psi(T_j)=\tilde{T}_j;\quad 1\leqq j\leqq m, \tag{4.6.1}$$

$$\psi(X)\in\tilde{T}';\quad X\in\mathscr{X} \tag{4.6.2}$$

Then we have:

Theorem. *The following conditions are equivalent:*

(i) ψ *is the restriction to* $\mathscr{X}$ *of an inner automorphism in* $\mathscr{L}\,(\mathscr{H})$;

(ii) *for any* z *in* D *there exists a unitary operator* $U_z:\mathscr{K}^{(n)}(T;\ z)\to\mathscr{K}^{(n)}(\tilde{T};\ z)$ *so that*

$$\psi\,(X)\,|\mathscr{K}^{(n)}(\tilde{T};\ z)=U_zXU_z^*|\mathscr{K}^{(n)}(\tilde{T};\ z);\quad X\in\mathscr{X}. \tag{4.6.3}$$

Proof. It sufficies to prove that, under our assumptions, the present condition (ii) implies the condition (ii) in Theorem 3.6.

Let $p=[f_T]$ and $\tilde{p}=[f_{\tilde{T}}]$. From Proposition 4.5. we obtain

$$U_zD_Ip\,(z)\ U_z^*=D_I\tilde{p}\,(z);\quad z\in D,\quad 0\leqq|I|\leqq n. \tag{4.6.4}$$

Now let us put $V_z=U_zp\,(z)$. By (4.6.3) and (4.6.4) we derive easily the desired relations (3.6.1) and (3.6.2). This concludes the proof.

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