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A Uniqueness Result for QVIs in Vector Lattices

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The uniqueness of certain sub-optimal controls in the sense of Lions is established by means of an abstract result for quasi-variational inequalities.

1. Let D be a bounded open subset of R^N with the smooth boundary Γ and $H^1(D)$ the usual Sobolev space. We consider the second order partial differential operator A from $H^1(D)$ into its dual $H^1(D)'$ defined by

$$\langle Au,\ w\rangle=\int_D[\sum_{i,j=1}^{N} a_{ij}(x)D^iu\ D^jw+a_0(x)uw+\sum_{j=1}^{N}(a_j(x)wD^ju+b_j(x)uD^jw)]\,dx,$$

with the coefficients $a_{ij}\in L^\infty(D)$, a_j, $b_j\in L^r(D)$, $a_0\in L^{r/2}(D)$ where $r\geqq N>2$ or $r>N=2$. Given a mapping $M: H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$, $f\in L^2(D)$ and $c>0$, we consider also the control problem

$$\min_{\substack{v\in L^2(\Gamma)\\ 0\leqq v\leqq c}} \int_\Gamma |y(v)-My\ (v)\ |^2\, d\Gamma,$$

where the state $y(v)$ is given by $Ay=f$ (in the sense of distributions on D) and $dy/dn=v$ in $H^{-1/2}(\Gamma)$. Asub-optimal control v^* in the sense of J. L. Lions ([5], Ch.3) is then $v^*=du/dn$, where u is a solution for

$$(1)\qquad \langle Au,w-u\rangle+c\int_\Gamma(Mu-w)^+\,d\Gamma\geqq\int_D f(w-u)\,dx$$
$$+c\int(Mu-u)^+\,d\Gamma\quad\forall\ \ w\in H(D).$$

The problem (1) is a quasi-variational inequality (QVI) in the sense of Tarţar — Joly — Mosco (see e. g. [6] or [10]). The existence of solutions for (1) follows by our abstract results from [8] [9], provided M has an adequate continuity property and

$$(2)\qquad \min\{\alpha,\ a\}\geqq c\sum_j(\|a_j\|_{L^r(D)}+\|b_j\|_{L^r(D)}),$$

where c, α, a are such that $\|u\|_{L^{\frac{2r}{r-2}}(D)}\leqq c\|u\|_{H^1(D)}$ (Sobolev imbedding)

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqq \alpha \sum_j \xi_j^2, \text{ a. e. in } D, \ \forall\ \xi\in R^N$$

(uniform ellipticity) and $a_0(x)\geqq a>0$ for a. e. $x\in D$.

Remark 1. In [5]-Ch. 3 the solution of (1) is constructed by an iterative procedure in the particular case $f\geqq 0$,

$$\langle Au,\ w\rangle=\int_D \nabla u\,\nabla w\,dx \quad\text{and}\quad Mu=(1/\text{meas}\ \Gamma)\int_\Gamma u\,d\Gamma.$$

Using the fact that in this situation Mu is constant on Γ, the solution can eventually be found as a fixed point of a real selection map of real variable (like for QVIs with implicit obstacle in [1]-Ch. IV §7).

In the sequel we are interested in the uniqueness of the solution for QVIs of type (1).

2. We state first an abstract criterion that assures that two solutions u_1, u_2 of a QVI are comparable $u_1\leqq u_2$. The framework is that of a vector lattice.

More precisely let us consider L a real linear space and P a convex cone in E such that $P\cap(-P)=\{0\}$; P induces an ordering on E by setting $v_1\leqq v_2$ (for $v_1, v_2\in E$) if and only if $v_2-v_1\in P$. Suppose that for all $v_1, v_2\in E$ there is an element $\inf\{v_1, v_2\}=\inf\{v_2, v_1\}\in E$ such that $v_1\leqq v_2\Rightarrow\inf\{v_1, v_2\}=v_1$

$$(3)\qquad v_1-\inf\{v_1, v_2\}\in P,\quad v_1+\inf\{0, v_2-v_1\}=\inf\{v_1, v_2\}.$$

Denote $v^+=-\inf\{-v, 0\}$, $v^-=-\inf\{v, 0\}$ and $\sup\{v_1, v_2\}=-\inf\{-v_1, -v_2\}$.

Theorem 1. *Let A be a linear operator from E to its algebraic dual E', h a real function on $E\times E$, f a linear functional from E' and assume that*

$$(4)\qquad \langle Av^+,\ v^-\rangle=0,\quad \forall\ v\in E,$$

$$(5)\qquad \langle Av,\ v\rangle\geqq 0,\quad \forall\ v\in E \ \text{ and } \ \langle Av,\ v\rangle=0\Rightarrow v=0.$$

The solutions u_1, u_2 of the QVI $\langle Au,\ w-u\rangle+h(u,\ w)\geqq f(w-u)+h(u,\ u)$, $\forall\ w\in E$, verify $u_1\leqq u_2$ if and only if

$$(6)\qquad h(u_1,\ \inf\{u_1,\ u_2\})+h(u_2,\ \sup\{u_1,\ u_2\}\leqq h(u_1,\ u_1)+h(u_2,\ u_2).$$

Proof. The necessity is immediate. Conversely, if u_1, u_2 are solutions, then the QVI for u_1 and $w=u_1+\inf\{0,\ u_2-u_1\}=\inf\{u_1,\ u_2\}$ gives

$$\langle Au_1,\ \inf\{0,\ u_2-u_1\}\rangle+h(u_1,\ \inf\{u_1,\ u_2\})\geqq f(\inf\{0,\ u_2-u_1\})+h(u_1,\ u_1).$$

The same QVI written for u_2 and $w=u_2-\inf\{0, u_2-u_1\}=-[-u_2+\inf\{0, -u_1-(-u_2)\}]=-\inf\{-u_2,\ -u_1\}=-\sup\{u_1,\ u_2\}$ is

$$\langle Au_2,\ -\inf\{0,\ u_2-u_1\}\rangle+h(u_2,\ \sup\{u_1,\ u_2\})\geqq -f(\inf\{0,\ u_2-u_1\})+h(u_2,\ u_2).$$

We add these two inequalities and we find

$$\langle A(u_2-u_1),\ -\inf\{0,\ u_2-u_1\}\rangle+h(u_1,\ \inf\{u_1,\ u_2\})+h(u_2,\ \sup\{u_1,\ u_2\}$$
$$\geqq h(u_1,\ u_1)+h(u_2,\ u_2).$$

Due to (3), one can easily infer that $u_2-u_1=(u_2-u_1)^+-(u_2-u_1)^-$. It follows that

$$\langle A(u_2-u_1)^+,\ (u_2-u_1)^-\rangle-\langle A(u_2-u_1)^-,\ (u_2-u_1)^-\rangle+h(u_1,\ \inf\{u_1,\ u_2\})$$
$$+h(u_2,\ \sup\{u_1,\ u_2\})+h(u_1,\ u_1)+h(u_2,\ u_2)$$

and using (4) one gets

$$\langle A(u_2-u_1)^-,\ (u_2-u_1)^-\rangle\leqq h(u_1,\ \inf\{u_1,\ u_2\})+h(u_2,\ \sup\{u_1,\ u_2\})$$
$$-h(u_1,\ u_1)-h(u_2,\ u_2);$$

hence, by (6) we must have $\langle A(u_2-u_1)^-,\ (u_2-u_1)^-\rangle\leqq 0$.
Then the positivity property (5) implies $(u_2-u_1)^-=0$ and finally one has

$$u_2-u_1=u_2-u_1+(u_2-u_1)^-=u_2-u_1-\inf\{u_2-u_1,\ 0\}\in P.$$

Remark 2. Obviously, if the property (6) stands $\forall\ u_1, u_2\in E$ then the QVI has at most one solution.

Remark 3. The assumptions on the order of E are in fact weaker* than those which characterize the structure of a vector lattice ([2]-Cap. I. §4). It should also be emphasized that the uniqueness of the solution follows on the basis of (3) (4) (5) (6) without assuming any topological structure on E, but (6) is available in the concrete case from Theorem 2 through an indirect compatibility between the Banach structure and the lattice structure.

3. We return to the concrete QVI (1) which characterizes suboptimal controls. Let us state

Theorem 2. *If (2) is valid for the QVI (1), if the mapping $M: H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$ is locally decreasing i. e.*

$$\text{(7)}\quad \left|\begin{array}{l}\text{for } v_1,\ v_2\in H^{1/2}(\Gamma) \text{ and } \Gamma_0=\{x\in\Gamma;\ v_1(x)\leqq v_2(x)\}\\ \qquad\text{one has } Mv_1\geqq Mv_2 \text{ on } \Gamma_0,\end{array}\right.$$

then the QVI (1) *admits at most one solution.*

Proof. Actually we want to apply Th. 1 (see Remark 2) taking $h(u,\ w)=c\int_\Gamma (Mu-w)^+\ d\Gamma$.

We remind that the natural a.e. order from $L^2(D)$(or$L^2(\Gamma)$) induces a vector sublattice structure on $H^1(D)$ (respectively on $H^{1/2}(\Gamma)$), so that (3) stands for $E=H^1(D)$. The convex cone P of the a. e. nonnegative elements of $H^1(D)$ verifies $P\cap(-P)=\{0\}$ and the differential operator from (1) verifies (4); to check (5) we use the hypothesis (2);

* The assumptions mean just that P is generating.

$$\langle Av, v\rangle \geqq \alpha \int_D \sum_j (D^j v)^2 \, dx + a \int_D v^2 \, dx - \sum_j (\| a_j \|_{L^r} + \| b_j \|_{L^r}) \| D^j v \|_{L^2} \| v \|_{L^{2r/(r-2)}}$$

$$\geqq \min \{\alpha, a\} \| v \|_{H^1}^2 - c \sum_j (\| a_j \|_{L^r} + \| b_j \|_{L^r}) \| D^j v \|_{L^2} \| v \|_{H^1}$$

$$= [\mathrm{Min} \{\alpha, a\} - c \sum_j (\| a_j \|_{L^r} + \| b_j \|_{L^r})] \| v \|_{H^1(D)}^2.$$

In this situation we must also examine the condition (6):

$$\text{(8)} \qquad \int_\Gamma (Mu_1 - \inf \{u_1, u_2\})^+ d\Gamma + \int_\Gamma (Mu_2 - \sup \{u_1, u_2\}^+ d\Gamma$$
$$\leqq \int_\Gamma (Mu_1 - u_1)^+ d\Gamma + \int_\Gamma (Mu_2 - u_2)^+ d\Gamma, \quad \forall \, u_1, u_2 \in H^1(D).$$

Let us prove first that the trace operator from $H^1(D)$ onto $H^{1/2}(\Gamma)$ is a lattice homomorphism ([7], Ch. II 3), that is

$$\text{(9)} \qquad \begin{aligned} (\inf \{u_1, u_2\})|_\Gamma &= \inf \{u_1|_\Gamma, u_2|_\Gamma\} \\ (\sup \{u_1, u_2\})|_\Gamma &= \sup \{u_1|_\Gamma, u_2|_\Gamma\}. \end{aligned}$$

$C_0^\infty(\bar{D})$ being dense in $H^1(D)$ one has $\varphi_n \to u_1$, $\psi_n \to u_2$ for some $\varphi_n, \psi_n \in C_0^\infty(\bar{D})$ and clearly

$$\text{(10)} \qquad (\inf \{\varphi_n, \psi_n\})|_\Gamma = \inf \{\varphi_n|_\Gamma, \psi_n|_\Gamma\}, \quad \forall \, n.$$

We can pass to the limit in (10) strongly in $L^2(\Gamma)$. Indeed, $\inf \{\varphi_n, \psi_n\}$ is bounded in $H^1(D)$ (by $\| \varphi_n \|_{H^1} + \| \psi_n \|_{H^1}$), hence it has a subsequence which converges weakly in $H^1(D)$. Thus in the first member of (10) we have a weak convergence in $H^{1/2}(\Gamma)$, which implies the convergence in $L^2(\Gamma)$. In the second member of (10), the corresponding subsequences of $\varphi_n|_\Gamma$, $\psi_n|_\Gamma$ converge in $L^2(\Gamma)$ and since $L^2(\Gamma)$ is a Banach lattice, "inf" is conveniently continuous; thus, for the second part of (10), we have convergence in $L^2(\Gamma)$, too. Hence (9) follows from (10) (note that the proof for "sup" is similar).

Next, due to (9) the inequality (8) becomes

$$\int_\Gamma [(Mv_1 - \inf \{v_1, v_2\})^+ + (Mv_2 - \sup \{v_1, v_2)^+] \, d\Gamma \leqq \int_\Gamma [(Mv_1 - v_1)^+$$
$$+ (Mv_2 - v_2)^+] \, d\Gamma, \quad \forall \, v_1, v_2 \in H^{1/2}(\Gamma).$$

In the region of Γ in which $v_1 \leqq v_2$ the integrands from both members of this inequality are the same.

The region of Γ in which $v_1 \geqq v_2$, taking into account the hypothesis (7), can be divided in six parts, respectively with: $Mv_1 \leqq Mv_2 \leqq v_2 \leqq v_1$, $Mv_1 \leqq v_2 \leqq Mv_2 \leqq v_1$, $v_2 \leqq Mv_1 \leqq Mv_2 \leqq v_1$, etc. One can elementarily verify that on each of them the first integrand is less than the integrand from the second member of the inequality.

Remark 4. The special uniqueness results for QVIs with implicit constraints from [1]-Ch. IV § 1 or [5]-Ch. 8 seem to be useless in this case, the same appears to be true concerning the uniqueness results from [3, 6] which assume the monotony and concavity of a selection map.

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