Mathematatica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica
Balkanica

New Series Vol. 2, 1988, Fasc. 1

A Comparison Theorem in the Theory of Quadrature Formulas

Geno P. Nikolov

Presented by Bl. Sendov

We prove here that the best quadrature formula in $W^r_q[a, b]$ which uses lower derivatives at a and b has a smaller error.

1. Introduction

Let $\bar\lambda=\{\lambda_1, \dots, \lambda_m\}$ and $\bar\mu=\{\mu_1, \dots, \mu_n\}$ be two sets of integers, such that $m+n\geqq r$,

$$0\leqq\lambda_1<\dots<\lambda_m\leqq r-1, \qquad 0\leqq\mu_1<\dots<\mu_n\leqq r-1,$$

where r is a given natural number.

Denote by $\Sigma_{\bar\lambda,\bar\mu}$ the class of all quadrature formulas of the form

$$(1)\qquad L\colon I(f)\approx L(f):=\sum_{k=1}^{m} A_k\, f^{(\lambda_k)}(a)+\sum_{k=1}^{n} B_k\, f^{(\mu_k)}(b),$$

where $I(f):=\int_a^b f(t)dt$ and the coefficients A_k and B_k are free parameters. We shall consider here these formulas in the class

$$B(W^r_q):=\{f\in C^{r-1}[a, b],\ f^{(r)}-\text{abs.cont. in } [a, b],\ \|f^{(r)}\|_{L_q[a,b]}\leqq 1\}, \qquad 1\leqq q\leqq\infty.$$

The error $R(L)$ of the quadrature formula (1) in $B(W^r_q)$ is defined as follows:

$$R(L):=\sup_{f\in B(W^r_q)}|I(f)-L(f)|.$$

Definition. *The quadrature formula $L^*\in\Sigma_{\bar\lambda,\bar\mu}$ is said to be the "best" in $\Sigma_{\bar\lambda,\bar\mu}$, if*

$$R(L^*)=\inf_{L\in\Sigma_{\bar\lambda,\bar\mu}} R(L):=R_{\bar\lambda,\bar\mu}.$$

We shall study only the classes $\Sigma_{\bar{\lambda},\bar{\mu}}$ for which $R_{\bar{\lambda},\bar{\mu}}<\infty$.

Given m and n, we say $(\bar{\lambda}', \bar{\mu}')\leqq(\bar{\lambda}, \bar{\mu})$, if

$$(2) \qquad \lambda_i'\leqq\lambda_i, \quad i=1, \dots, m, \quad \mu_j'\leqq\mu_j, \quad j=1,\dots, n.$$

Respectively, $(\bar{\lambda}', \bar{\mu}')<(\bar{\lambda}, \bar{\mu})$, if at least one of the inequalities (2) is strict.

The main result of this paper is the following:

Theorem. *Let* $(\bar{\lambda}', \bar{\mu}')<(\bar{\lambda}, \bar{\mu})$. *Then* $R_{\bar{\lambda}',\bar{\mu}'}< R_{\bar{\lambda}\bar{\mu}}$.

2. Auxiliary results

It is easy to see, that the best quadrature formula in $\Sigma_{\bar{\lambda},\bar{\mu}}$ is exact for all algebraic polynomials of degree $r-1$ (see f.e. [3]).

There is an important one-to-one correspondence (see [1]) between quadrature formulas

$$(3) \qquad I(f)\approx \sum_{k=1}^{m} A_k\, f^{(\lambda_k)}(a)+\sum_{k=1}^{n} B_k\, f^{(\mu_k)}(b)+\sum_{k=1}^{s} a_k\, f(x_k),$$

which are exact for all algebraic polynomials of degree $r-1$ and monosplines of degree r with knots $(x_i)_1^s$, $a<x_1<\dots<x_s<b$. In the case (1) the corresponding monosplines are simply algebraic polynomials of the form $P(t)=(-t)^r/r!+c_1t^{r-1}+\dots+c_r$, satisfying the boundary conditions

$$(4) \qquad \begin{aligned} &P^{(r-k-1)}(a)=0, \quad k\in\{0,\dots, r-1\}\setminus\bar{\lambda} \\ &P^{(r-k-1)}(b)=0, \quad k\in\{0,\dots, r-1\}\setminus\bar{\mu}. \end{aligned}$$

Moreover, $R(L)=\| P\|_p$, where $\|f\|_p:=\{\int_a^b|f(t)|^p dt\}^{1/p}$ and p is the conjugate number to q, i.e., $1/p+1/q=1$. Therefore,

$$(5) \qquad R_{\bar{\lambda},\bar{\mu}}=\frac{1}{r!}\inf_{P\in \mathcal{P}_{\bar{\lambda},\bar{\mu}}}\| P\|_p,$$

where $\mathcal{P}_{\bar{\lambda},\bar{\mu}}$ is the subset of those algebraic polynomials of degree r, which satisfy (4) and have a leading coefficient equal to 1.

Next we prove some auxiliary results from which the theorem easily follows.

Lemma 1. *Let* ξ_i, ν_i, $i=1, \dots, m$ *be integers satisfying the inequalities*

$$(6) \qquad \begin{aligned} &\nu_1>\nu_2>\dots>\nu_m\geqq 0, \quad \xi_1>\dots>\xi_m\geqq 0, \\ &\nu_i\geqq\xi_i, \quad i=1,\dots, m. \end{aligned}$$

Let $D=(d_{ij})_{m\times m}$, *where* $d_{ij}:=\nu_j(\nu_j-1)\cdot\ldots\cdot(\nu_j-\xi_i+1)$.

Then $\det D>0$.

Proof. We apply induction on m. For $m=2$ the assertion is evidently true. Next, we assume that it is true for each $k<m$. Note that, if there exists an integer l, $2\leqq l\leqq m$, such that $v_l<\xi_{l-1}$, then $d_{ij}=0$ for $i\leqq l-1, j\geqq l$, and thus D is of the form

$$\begin{pmatrix} D' & 0 \\ D'' & D''' \end{pmatrix},$$

where D', D''' are of lower dimension. Then $\det D=\det D' \ \det D'''>0$ in virtue of inductional hypothesis.

Now suppose that $v_l\geqq\xi_{l-1}$ for all $l=2,\dots,m$. We shall prove the statement for m by induction with respect to the number G of the gaps in the sequence $\xi_1,\dots,\xi_m$. For $G=0$ the statement is true because the functions $t^{v_1-\xi_m},\dots,t^{v_m-\xi_m}$ form an ET-system in $(0,\infty)$. Suppose that the assertion is true for $G-1$ gaps.

Denote
$$D\begin{pmatrix} i_1,\dots, i_p \\ j_1,\dots, j_p \end{pmatrix}:=\det\,(d_{i_\nu j_\mu})_{\nu=1,\,\mu=1}^{p\ \ \ p}.$$

By a basic identity for determinants we have

$$D\begin{pmatrix} 2,\dots, & m-1, & m+1 \\ 1, & \dots, & m-1 \end{pmatrix}\cdot D\begin{pmatrix} 1,\dots, & m \\ 1,\dots, & m \end{pmatrix}=D\begin{pmatrix} 2,\dots, & m \\ 1,\dots, & m-1 \end{pmatrix}$$

$$\cdot D\begin{pmatrix} 1,\dots, & m-1, & m+1 \\ 1,\dots, & & m \end{pmatrix}-D\begin{pmatrix} 1,\dots, & m-1 \\ 1,\dots, & m-1 \end{pmatrix}\cdot D\begin{pmatrix} 2,\dots, & m+1 \\ 1,\dots, & m \end{pmatrix}.$$

Now we use inductional hypothesis for $m-1$ dimension and for $G-1$ gaps, choosing ξ_{m+1} such that $\xi_s>\xi_{m+1}>\xi_{s+1}$, we obtain

$$\operatorname{sign} D\begin{pmatrix} 2,\dots, & m-1, & m+1 \\ 1, & \dots, & m-1 \end{pmatrix}=(-1)^{m-1-s},\quad \operatorname{sign} D\begin{pmatrix} 2,\dots, & m \\ 1,\dots, & m-1 \end{pmatrix}=1,$$

$$\operatorname{sign} D\begin{pmatrix} 1,\dots, & m-1, & m+1 \\ 1, & \dots, & m \end{pmatrix}=(-1)^{m-1-s},\quad \operatorname{sign} D\begin{pmatrix} 1,\dots, & m-1 \\ 1,\dots, & m-1 \end{pmatrix}=1,$$

$$\operatorname{sign} D\begin{pmatrix} 2,\dots, & m+1 \\ 1,\dots, & m \end{pmatrix}=(-1)^{m-s}.$$

Then $\operatorname{sign} D\begin{pmatrix} 1,\dots, & m \\ 1,\dots, & m \end{pmatrix}=1$, i.e., $\det D>0$. The lemma is proved.

Lemma 2. *Let ξ_i, v_i, $i=1,\dots,m$ be as in Lemma 1. Then there exists a unique polynomial $P(t)$ of the form*

$$P(t)=t^r+a_1t^{v_1}+\dots+a_mt^{v_m}, \tag{7}$$

satisfying the conditions $P^{(\xi_i)}(1)=0$, $i=1,\dots, m$. *In addition,*
(i) $P(t)$ *has no zeros in* $(0,1)$,
(ii) sign $a_k=(-1)^k$, $k=1,\dots, m$, *and, consequently,* sign $P(t)=(-1)^m$ *in* $(0,1)$.

Proof. The existence part and the uniqueness part follow from Lemma 1 and conditions

$$\sum_{j=1}^{m} \nu_j\cdot(\nu_j-1)\cdot\ldots\cdot(\nu_j-\xi_i+1)\cdot a_j=-r\cdot(r-1)\cdot\ldots\cdot(r-\xi_i+1):=-C_i,\ i=1,\dots, m.$$

By Krammer's formulas $a_k=(-1)^k \det\Delta_k/\det D$, $k=1,\dots, m$, where Δ_k is obtained from D, deleting the k-th column and putting $(c_1,\dots, c_m)^T$ as first. By Lemma 1 det $\Delta_k>0$. Hence, sign $a_k=(-1)^k$.

Since

$$P^{(i)}(0)=0,\ j\in\{0,\dots, r-1\}\setminus\{\nu_1,\dots, \nu_m\}$$

$$P^{(j)}(1)=0,\ j\in\{\xi_1,\dots, \xi_m\},$$

the assumption $P(t)=0$ for some $t\in(0, 1)$ holds by Rolle's theorem to the contradiction $0=P^{(r)}(x)$! Therefore, $P(t)\neq 0$ in $(0, 1)$. Then, sign $P(t)=$sign a_m $=(-1)^m$ in $(0, 1)$.

Lemma 3. *Suppose that* $m, s, (\nu_k)_1^m, (\xi_k)_1^{m-s}$ *are integers such that* $m>s>0$ *and* $r-1\geqq\nu_1>\ldots\nu_m\geqq 0$, $r-1\geqq\xi_1>\ldots>\xi_{m-s}\geqq 0$, $\nu_i\geqq\xi_i$, $i=1,\dots, m-s$.
Given $0<x_1\leqq\ldots\leqq x_s<1$, *there exists a unique polynomial* $P(t)$ *of the type* (7) *satisfying the conditions*

$$P^{(\xi_i)}(1)=0,\quad i=1,\dots, m-s, \tag{8}$$

which vanishes at $(x_i)_1^s$ *(under the condition that* $P(x_i)=P'(x_i)=\ldots=P^{(l)}(x_i)=0$, *if* $x_{i-1}<x_i=x_{i+1}=\ldots=x_{i+l}<x_{i+l+1}$).

Proof. Since the incidence matrix E corresponding to our interpolation problem does not contain supported blocks of one-entries, we need to show only (according to the Atkinson – Sharma theorem [2]) that E satisfies the Pólya condition: $M_k\geqq k+1$, where M_k is the number of one-entries in the first k column of E.

Clearly this holds for $k=1,\dots, s$, since $P(t)$ vanishes at $(x_i)_1^s$. Zeros in the first row of E correspond to $\nu_m,\dots, \nu_1$, and units in the last row of E correspond to the numbers $\xi_{m-s},\dots, \xi_1$.

The inequality $\xi_{m-s}\leqq\nu_{m-s}$ shows that the first unit in the last row of E appears before $s+1$-th zero in the first row of E. Next, it is not difficult to verify that the conditions $\xi_i \leqq \nu_i$, $i=m-s-1,\dots,1$, imply $M_k \geqq k+1$ for $k=s+1,\dots,r$. The lemma is proved.

Remark 1. It follows from Lemma 3 that the coefficients $(a_k)_1^m$ of the polynomial $P(t)$ are continuous functions of the points $(x_i)_1^s$, $0 \leqq x_1 \leqq \dots \leqq x_s \leqq 1$.

Remark 2. Clearly the incidence matrix E from Lemma 3 will remain still poised, if we delete several last columns, if the number of one-entries in the new matrix is equal to the number of columns of this matrix.

Lemma 4. *Let* $a=0$, $b=1$, $m+n=r+s$, $s>0$, *and let* $P(t) \in \Sigma_{\bar{\lambda},\bar{\mu}}$ *be the polynomial, which corresponds to the best quadrature formula in* $\Sigma_{\bar{\lambda},\bar{\mu}}$. *Then* $P(t)$ *has exactly* s *zeros in* $(0, 1)$ *(counting multiplicities).*

Proof. According to the conditions (4) the polynomial $P(t)$ is of the type (7) and satisfies (8), where $\nu_j = r-1-\lambda_j$, $j=1,\dots,m$; $\{\xi_1 > \dots > \xi_{m-s}\} = \{0,\dots,r-1\} \setminus \{r-1-\mu_j,\ j=1,\dots,n\}$. It follows from Lemma 1 that $\nu_i \geqq \xi_i$, $i=1,\dots,m-s$. Evidently, $P(t)$ has no more than s zeros in $(0, 1)$. Otherwise, a repeated application of Rolle's theorem yields that $P^{(r)}(t)$ has a zero in $(0, 1)$, that is, a contradiction.

Suppose now that $P(t)$ has l zeros in $(0, 1)$, $l<s$.

Case A. $s-l-1$ is an even number. Let construct a polynomial $q(t)=b_1 t^{\nu_1} + \dots + b_m t^{\nu_m}$ satisfying (8), which vanishes at $(x_i)_1^l$ and, in addition, has a zero of multiplicity $s-l-1$ at some point $x_{l+1} \in (0, 1)$, $x_{l+1} \neq x_i$, $i=1,\dots,l$. According to Remark 2, such a polynomial exists. Moreover, $q(t)$ has no other zeros in $(0, 1)$. Define

$$Q_\varepsilon(t) := P(t) - \varepsilon\ q(t),$$

where ε is a real number, such that

$$\varepsilon\ P(t)\ q(t) \geqq 0, \quad |P(t) \geqq |\varepsilon\ q(t)| \quad \text{in } (0,\ 1).$$

Then, $|Q_\varepsilon(t)| \leqq |P(t)|$ in $(0, 1)$. Therefore, $\| Q_\varepsilon \|_p < \| P \|_p$. On the other hand, by definition $\| P \|_p = \inf \{ \| Q \|_p : Q \in \mathscr{P}_{\bar{\lambda},\bar{\mu}} \}$. Since $Q_\varepsilon \in \mathscr{P}_{\bar{\lambda},\bar{\mu}}$, we get a contradiction.

Case B. $s-l-1$ is an odd number. As in the previous case we construct a polynomial

(i) $q(t) = b_1 t^{\nu_{k+1}} + \dots + b_{m-k} t^{\nu_m}$ with additional zero at $x_{l+1} \in (0, 1)$ of implicity $s-l-2$, if there is a k, $1 \leqq k \leqq m-s$, such that $\nu_1 \geqq \xi_1 > \nu_2 \geqq \xi_2 > \dots > \xi_k$, $\nu_{k+1} \geqq \xi_k$,

(ii) $q(t) = b_1 t^{\nu}_{m-s+1} \dots + b_s t^{\nu}_m$ with additional zero at $x_{l+2} \in (0, 1)$ of multiplicity $s-l-2$ and zero $x_{s-1} = 1$, if there is no such k.

Then, in the same way as in Case A, constructing a polynomial Q_ε, we get a contradiction.

Therefore, $P(t)$ has exactly s zeros in $(0, 1)$. The lemma is proved.

3. Proof of the theorem

Let for convenience $a=0$, $b=1$. First, we shall prove the theorem in the case $\bar{\mu}'=\bar{\mu}$, and $\bar{\lambda}'$ differs from $\bar{\lambda}$ by one element, i. e., $\bar{\lambda}'$ is obtained from $\bar{\lambda}$ by replacing some λ_k by α, where $\lambda_{k-1}<\alpha<\lambda_k$.

Let $m+n=r+s$. Every polynomial $P(t)\in\mathscr{P}_{\bar{\lambda},\bar{\mu}}$ is of the form (7) and satisfies (8), where $v_i=r-1-\lambda_i$, $\{\xi_1>\dots>\xi_{m-s}\}=\{0,\dots,r-1\}\setminus\{r-1-\mu_i, i=1,\dots n\}$. We suppose that $\xi_i\leqq v_i$, $i=1,\dots,m-s$. Otherwise, it follows easily from Lemma 1 that $\mathscr{P}_{\bar{\lambda},\bar{\mu}}$ is an empty set.

The proof goes by induction on s. Suppose that $s=0$. In this case there exist unique polynomials $P\in\mathscr{P}_{\bar{\lambda},\bar{\mu}}$ and $Q\in\mathscr{P}_{\bar{\lambda}',\bar{\mu}'}$. Let $P(t)=t^r+\sum_{i=1}^{m}a_it^{v_i}$. Consider the polynomial $q(t)=P(t)-Q(t)$. Clearly

$$q(t)=c_1t^{v_1}+\dots+c_{k-1}t^{v_{k-1}}+c_k\cdot t^{v}+a_k\cdot t^{v_k}+c_{k+1}t^{v_{k+1}}+\dots+c_m\cdot t^{v_m},$$

where $v=r-1-\alpha$ and $(c_j)_1^m$ are real numbers satisfying

$$\sum_{\substack{j=1\\ j\neq k}}^{m} v_j\cdot(v_j-1)\cdot\ldots\cdot(v_j-\xi_i+1)\cdot c_j+v\cdot(v-1)\cdot\ldots\cdot(v-\xi_i+1)c_k$$

$$=-a_kv_k\cdot(v_k-1)\cdot\ldots\cdot(v_k-\xi_i+1)\qquad i=1,\dots, m.$$

By Lemma 1 the determinant of the coefficients of this system is non-zero. As in Lemma 2, we get

$$\operatorname{sign} c_j=-\operatorname{sign} a_k\cdot(-1)^{j-k-1}=(-1)^j,\quad j=1,\dots, m.$$

It follows, on the basis of Remark 2, that $q(t)$ has no zeros in (0, 1). Indeed, otherwise $q(t)\equiv0$, and hence, $a_k=0$. But, as we show in Lemma 2, sign $a_k=(-1)^k$. Thus, $q(t)\neq0$ in (0, 1). Therefore,

$$\text{(10)}\qquad \operatorname{sign} P(t)=\operatorname{sign} Q(t)=\operatorname{sign}\,[P(t)-Q(t)]$$

in (0, 1), which yields $|P(t)|>|Q(t)|$ in (0, 1) and consequently, $\| P\|_p>\| Q\|_p$, which had to be shown.

Now, let $0<s\leqq r-1$. Suppose that the theorem is true for each choice of m and n such that $0\leqq m+n-r<s$. With each $P\in\mathscr{P}_{\bar{\lambda},\bar{\mu}}$ having s zeros (say $\tau_1\leqq\dots\leqq\tau_s$) in (0, 1) we associate the polynomial $Q\in\mathscr{P}_{\bar{\lambda}',\bar{\mu}'}$, which has the same zeros $(\tau_i)_1^s$.

Our next goal is to show that (10) is true in this case. Evidently, (10) implies $\| Q\|_p<\| P\|_p$ and thus, the assertion of the theorem.

We have already proved (10) for $s=0$. Suppose that (10) holds for each choice of m and n such that $0 \leqq m+n-r<s$. Assume that $m+n-r=s$. It follows from Lemma 3 that there exist unique polynomials $P \in \mathscr{P}_{\bar{\lambda},\bar{\mu}}$ and $Q \in \mathscr{P}_{\bar{\lambda}',\bar{\mu}'}$, with given zeros $\tau_1 \leqq \ldots \leqq \tau_s$ in (0, 1). Evidently, $P(t)$ and $Q(t)$ have no other zeros in (0, 1) except $(\tau_i)_1^s$. Let $P(t)$ be of the form

$$P(t)=t^r+a_1 t^{\nu_1}+\ldots+a_m t^{\nu_m}.$$

Then the polynomial $q(t)=P(t)-Q(t)$ has the form

$$q(t)=\sum_{\substack{i=1 \\ i \neq k}}^{m} c_i t^{\nu_i}+c_k . t^{\nu}+a_k t^{\nu_k}.$$

It is seen that $a_i \neq 0$, $i=1, \ldots, m$ (otherwise we will get $P^{(r)}(t)=0$ i.e., a contradiction).

Now, we can show that $q(t)$ has no other zeros in (0, 1) except $(\tau_i)_1^s$. Really, $q(t)$ satisfies the boundary conditions

$$\begin{aligned} & q^{(r-i-1)}(0)=0, \quad i \in\{0, \ldots, r-1\} \backslash\{\bar{\lambda}, \alpha\}; \\ (11) \qquad & q^{(r-i-1)}(1)=0, \quad i \in\{0, \ldots, r-1\} \backslash \bar{\mu}; \\ & q^{(r-\lambda_k-1)}(0)=P^{(r-\lambda_k-1)}(0). \end{aligned}$$

If we suppose that q has $s+1$ zeros in (0, 1), then (11) yields (by Rolle's theorem) that $q(t) \equiv 0$, which is inconsistent with $a_k \neq 0$.

Now, let τ_s tend to 1. Then P and Q will tend uniformly to some polynomials P_0 and Q_0 from $\mathscr{P}_{\bar{\lambda},\bar{\mu}_0}$ and $\mathscr{P}_{\bar{\lambda}',\bar{\mu}_0'}$, respectively, where $\bar{\mu}_0$ is obtained by "adding" the new boundary condition $P(1)=0$ to the boundary conditions of $\mathscr{P}_{\bar{\lambda},\bar{\mu}}$ and $\mathscr{P}_{\bar{\lambda}',\bar{\mu}'}$. Clearly, the parameters m and n_0, corresponding to $\bar{\lambda}$, $\bar{\lambda}'$ and $\bar{\mu}_0 \equiv \bar{\mu}_0'$, satisfy the inequality $m+n_0-r \leqq s-1$ and by the inductive hypothesis

$$\operatorname{sign} P_0(t)=\operatorname{sign} Q_0(t)=\operatorname{sign}\left[P_0(t)-Q_0(t)\right] \text { in }(0,1).$$

The coefficients $(a_i)_1^m$, $(b_i)_1^m$, $(c_i)_1^m$ of P, Q and q, respectively, are continuous functions of $(\tau_i)_1^s$ in $0 \leqq \tau_1 \leqq \ldots \leqq \tau_s \leqq 1$. Then, a_m, b_m, c_m do not change signs when τ_s moves to 1, otherwise they vanish for some $\tau_s \in[\tau_{s-1}, 1]$. Thus (10) holds.

So we get a polynomial $Q \in \mathscr{P}_{\bar{\lambda}',\bar{\mu}'}$ with a smaller L_p-norm than P. Therefore,

$$(12) \qquad R_{\bar{\lambda}',\bar{\mu}'}<R_{\bar{\lambda},\bar{\mu}}.$$

Now the general case follows by finite number of pair-wise comparisons of type (12). The theorem is proved.

4. Corollaries

We mention here some consequences of the main result.

Corollary 1. *Let* $1 \leqq q \leqq \infty$. *The best quadrature formula of the form*

$$I(f) \approx \sum_{k=0}^{m-1} A_k f^{(k)}(a) + \sum_{k=0}^{n-1} B_k f^{(k)}(b)$$

has a minimal error $R_{m,n}$ *in* $B(W_q^r)$ *amidst all formulas of the type* (1). *Moreover,*

$$R_{m,n} = \frac{1}{r!} \inf_{c_1,\ldots,c_s} \{\int_a^b |(t-a)^{r-m}(t-b)^{r-n}(t^s + c_1 t^{s-1} + \ldots + c_s)|^p dt\}^{1/p}.$$

Remark 3. In the case $m+n=r$ the classical Tchakalov — Obreshkov formula is extremal.

Corollary 2. *Let* $(v_i)_1^m$, $(\xi_i)_1^l$, $0 \leqq l < m$ *be given integers such that*

$$r-1 \geqq v_1 > \ldots > v_m \geqq 0, \quad r-1 \geqq \xi_1 > \ldots > \xi_l \geqq 0, \quad v_i \gneqq \xi_i, \; i=1,\ldots, l.$$

Let $E(\bar{v}) = \inf_{a_1,\ldots,a_m} \{\|P\|_{L_i[0,1]} : P(t) = t^r + a_1 t^{v_1} + \ldots + a_m t^{v_m}, \; P^{(\xi_i)}(1) = 0, \; i=1,\ldots, l\}$.
Then $E(\bar{v}) = E(v_1,\ldots, v_m)$ *is a decreasing function of* $v_1,\ldots, v_m$.

Note that in the special case $l=0$ (i.e., there are no boundary conditions on $P(t)$ at 1) we give an answer to the well-known non-linear problem of G. G. Lorentz about the approximation of t^r by linear combination of m functions $(m<r)$ amidst $\{t^0,\ldots, t^{r-1}\}$ (see [4]).

The author wishes to express his thanks to B. Bojanov for his constant attention and aid on the topic.

References

1. B. D. Bojanov. Existence and characterization of monosplines of least L_p deviation. — In: Constructive function theory '77. Sofia, 1980, 249-268.
2. K. Atkinson, A. Sharma. A partial characterization of poised Hermite — Birkhoff interpolation problems. *SIAM J. Numer. Anal.*, **6**, 1969, 230-235.
3. S. M. Nikolski. Quadrature formulas. Moskow, 1958.
4. Ph. V. Smith. An improvement theorem for Descartes systems. *Proc. Amer. Math. Soc.*, **70**, 1959, 480-481.

Institute of Mathematics
P. O. Box 373
1090 Sofia,
BULGARIA

Received 09.07.1986