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Comparison of Recovery Schemes Based on End-Point Values

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We prove that the L_p-norm of the perfect spline satisfying some zero boundary conditions

$$\mathscr{B}:=\{f^{(j)}(a)=0,\quad j\in J_1,\quad f^{(j)}(b)=0,\quad j\in J_2\},$$

and having a maximal number of fixed zeros in (a, b), is a decreasing function of the order of the derivatives used in $\mathscr{B}$. This fact has a nice interpretation in the theory of optimal recovery.

1. Introduction

This paper is concerned with the best approximation of functions f from a given class W on the basis of some information data $T(f)$. In our study $T(f)$ consists of point evaluations of f or its derivatives, i. e.,

$$T(f):=\{l_1(f),\ldots,l_N(f)\},$$

where $\{l_k(f)\}$ are fixed functionals of the form $l_k(f)=f^{(j_k)}(x_k)$.

Any transformation $S:\mathrm{R}^N\to L_\infty[a, b]$ generates a recovery scheme S for functions f from W in the following way:

(1) $$f(x)\approx S(l_1(f),\ldots,l_N(f))(x),\quad x\in[a, b].$$

The error $R_S(f)$ of the approximation (1) is usually defined as L_p-norm of the difference $f-S$ for some fixed p. Set:

$$R_S:=\sup_{f\in W}\|f-S(l_1(f),\ldots,l_N(f))\|_p.$$

Next we recall the central notion in the theory of optimal recovery.

Definition. *The method S^* is said to be best method of recovery in the class W on the basis of the information $T(f)$ if*

$$R_{S^*}=\inf_S R_S=:R(T).$$

By a comparison theorem we mean a statement of the form: If $T_1<T_2$ (that is, if the information T_1 precedes T_2, according to some easily checked natural criteria), then $R(T_1)\leqq R(T_2)$.

Given $[a, b]$ we consider here the class

$$W=B(W^r_\infty):=\{f\in W^r_\infty[a, b]: \|f^{(r)}\|_\infty\leqq 1\},$$

where

$$W^r_\infty[a, b]:=\{f: f\in C^{r-1}[a, b],\quad f^{(r-1)}\text{abs. cont.},\quad \|f^{(r)}\|_\infty<\infty\}.$$

Our central result is Theorem 2, which shows that the error $R(T)$ of the best recovery scheme in $B(W^r_\infty)$ is a monotone function of the order of the derivatives at the end-points occurring in the information data T.

2. Preliminaries: perfect splines

We review in this section some basic properties of the well-known polynomial perfect splines. They proved to be a very useful technique in studying extremal problems in $W^r_\infty[a, b]$.

A perfect spline of degree r with knots $(\xi_k)_1^n$, $\xi_1<\ldots<\xi_n$, is every expression of the form

$$s(t)=\sum_{i=1}^{r}\alpha_i t^{i-1}+c\left[t^r+2\sum_{k=1}^{n}(-1)^k(t-\xi_k)^r_+\right],$$

where $\alpha_1,\ldots, \alpha_r$ and c are real parameters.

All propositions listed below are slight extensions of known facts (see [2]). They are proved in more general setting in [1].

Given the points $\bar{x}=(x_i)_1^n$, $a<x_1<\ldots<x_n<b$ with multiplicities $(\nu_i)_1^n$, respectively $(1\leqq\nu_i\leqq r)$, and the set $I:=(\bar{\lambda}, \bar{\mu})$ of integers

$$\{\lambda_1,\ldots, \lambda_{m_1}\}=:\bar{\lambda},\quad 0\leqq\lambda_1<\ldots<\lambda_{m_1}\leqq r-1,$$

$$\{\mu_1,\ldots, \mu_{m_2}\}=:\bar{\mu},\quad 0\leqq\mu_1<\ldots<\mu_{m_2}\leqq r-1,$$

we denote by $T(\bar{x}, I; f)$ the data

$$\{f^{(j)}(a),\ j\in\bar{\lambda};\quad f^{(j)}(b),\ j\in\bar{\mu};\quad f^{(j)}(x_i),\quad i=1,\dots,n,\quad j=0,\dots,\nu_i-1\}.$$

Set $N:=\nu_1+\dots+\nu_n+m_1+m_2$.

Theorem A. *Given the set $(\bar{x}, I)$, there exists a unique perfect spline $\varphi(\bar{x}, I; t)$ of degree r with no more than $N-r$ knots, such that $\|\varphi^{(r)}\|_\infty=1$, $(-1)^n c\geqq 0$, and*

$$T(\bar{x}, I; \varphi)=\bar{0}. \tag{2}$$

Moreover, $\varphi(\bar{x}, I; t)$ has exactly $N-r$ knots and no more zeros than those prescribed by (2).

The next theorem reveals a beautiful extremal property of the perfect splines.

Theorem B. *Let $(\bar{x}, I)$ be an arbitrary fixed set. Then.*

$$|f(t)|\leqq|\varphi(\bar{x}, I; t)| \quad on \quad [a, b] \tag{3}$$

for each function $f\in B(W^r_\infty)$, such that $T(\bar{x}, I; f)=\bar{0}$.

Now denote by $R_p(T)$, the L_p-error of the best recovery scheme in $B(W^r_\infty)$ on the basis of the information T. Then, one could easily derive from Theorem B (see [1]) that

$$\mathrm{R}_p(T(\bar{x}, I))=\|\varphi(\bar{x}, I; \cdot)\|_p. \tag{4}$$

Further we shall frequently refer to (4) in order to present some assertions about $\varphi(\bar{x}, I)$ as comparison theorems in $B(W^r_\infty)$.

Theorem C. *Let $(\xi_i)_1^M$ be the knots of $\varphi(\bar{x}, I; t)$, $M:=N-r$. Then $\varphi^{(r-1)}(\bar{x}, I; \xi_i)\neq 0$ for $i=1,\dots, M$.*

Proof. The assertion was actually proved in [1]. We sketch here the proof in order to make the reasoning in the next section clearer.

By Rolle's theorem, $\varphi^{(r-1)}(t)$ has exactly $M+1$ distinct zeros in $[a, b]$. Denote them by $(\eta_i)_1^{M+1}$. Thus, $\varphi^{(r-1)}(t)\neq 0$ if $t\notin\{\eta_1,\dots,\eta_{M+1}\}$. Since $\varphi^{(r)}(t)$ changes sign only at the knots $\{\xi_i\}$, it follows again by an extention of Rolle's theorem that $\eta_i<\xi_i<\eta_{i+1}$, $i=0,\dots, M$. Therefore, $\xi_i\notin\{\eta_1,\dots,\eta_{M+1}\}$, and consequently, $\varphi^{(r-1)}(\xi_i)\neq 0$.

3. Main results

The proof of our central theorem relies upon an estimation for the number of zeros of perfect splines.

Given a function $f \in C^{r-1}[a, b]$, we denote by $Z(f;\ (a, b))$ the number of zeros of f in (a, b) counting multiplicities up to order r.

Note here that according to Theorem C, the knots $\{\xi_i\}$ of the perfect spline $\varphi(\bar{x},\ I;\ t)$ could not be zeros of φ of multiplicity greater than $r-1$.

We use in this paper the customary notations $S^+(f_1, \ldots, f_m)$ and $S^-(f_1, \ldots, f_m)$ for the number of weak and, respectively, strong sign changes in the sequence $\{f_i\}_1^m$ (see [4] for details).

The following estimation could be recognized as Budan-Fourier theorem for perfect splines.

Theorem 1. *Let φ be an arbitrary perfect spline of degree r with M knots in (a, b). Then*

$$(5)\qquad Z(\varphi;\ (a, b)) \leqq M + S^-(\varphi(a),\ \varphi'(a), \ldots, \varphi^{(r)}(a)) - S^+(\varphi(b),\ \varphi'(b), \ldots, \varphi^{(r)}(b)).$$

Proof. According to the classical Budan-Fourier theorem for algebraic polynomials (see for example [4], Theorem 3.9),

$$(6)\quad Z(f;\ (\alpha, \beta)) \leqq S^-(f(\alpha), f'(\alpha), \ldots, f^{(r)}(\alpha)) - S^+(f(\beta), f'(\beta), \ldots, f^{(r)}(\beta))$$

for any polynomial f of degree r with non-zero leading coefficient and any finite interval (α, β).

In order to prove Theorem 1 we need only apply (6) for $(\alpha, \beta) = (\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, M$, where $\xi_0 := a$, $\xi_{M+1} := b$. We get

$$Z(\varphi;\ (a, b)) \leqq \sum_{i=1}^{M} \delta_i - \sum_{i=1}^{M} s_i + S^-(\varphi(a), \ldots, \varphi^{(r-1)}(a),\ (-1)^M) - S^+(\varphi(b), \ldots, \varphi^{(r-1)}(b),\ 1)$$

where δ_i is the multiplicity of the zero of φ at ξ_i and

$$s_i := S^+(\varphi(\xi_i), \ldots, \varphi^{(r-1)}(\xi_i),\ (-1)^{M-i-1}) - S^-(\varphi(\xi_i), \ldots, \varphi^{(r-1)}(\xi_i),\ (-1)^{M-i}).$$

Clearly, $\delta_i - s_i \leqq 1$ and the proof is completed.

We shall write $I_1 < I_2$ to indicate that $\lambda_k^{(1)} \leqq \lambda_k^{(2)}$, $\mu_k^{(1)} \leqq \mu_k^{(2)}$ for all k, with at leats one strict inequality, where $\lambda_k^{(i)}$, $\mu_k^{(i)}$ are the corresponding elements of I_i, $i = 1, 2$.

Now we are prepared to state and prove our main result.

Theorem 2. *Let $N+m_1+m_2>r$. Suppose that $I_1<I_2$. Then*

$$|\varphi(\bar{x}, I_1; t)|\leqq|\varphi(\bar{x}, I_2; t)| \quad on \quad [a, b] \tag{7}$$

with strict inequality on some subinterval.

Proof. Set $I=((\lambda_1,\dots, \lambda_{m_1}), (\mu_1,\dots, \mu_{m_2}))$ for simplicity. Let k $(0\leqq k\leqq r-1)$ be a fixed integer satisfying

$$\lambda_k+1<\lambda_{k+1} \text{ if } k<m_1, \ \lambda_k+1\leqq r-1 \text{ if } k=m_1.$$

Define the set $\hat{I}=((\hat{\lambda}_1,\dots, \hat{\lambda}_{m_1}), (\mu_1,\dots, \mu_{m_2}))$ in the following way:

$$\hat{\lambda}_i=\begin{cases}\lambda_i & \text{if} \quad i\neq k,\\ \lambda_i+1 & \text{if} \quad i=k.\end{cases}$$

Evidently, Theorem 2 will follow by pair-wise comparisons if we prove that

$$|\varphi(\bar{x}, I; t)|\leqq|\varphi(\bar{x}, \hat{I}; t)| \quad \text{on} \quad [a, b] \tag{8}$$

with strict inequality on some subinterval.

Our next goal is to prove (8).

Let us introduce the set $I_0:=(\bar{\lambda}_0, \bar{\mu})$, where $\bar{\lambda}_0=(\lambda_1,\dots, \lambda_{k-1}, \lambda_{k+1},\dots, \lambda_{m_1})$, $\bar{\mu}=(\mu_1,\dots, \mu_{m_2})$.

According to Theorem A, there exists a unique perfect spline $\varphi_0(t):=\varphi(\bar{x}, I_0; t)$ of degree r with $M-1$ knots, such that $T(\bar{x}, I_0; \varphi_0)=\bar{0}$.

Next we investigate the sign pattern of φ_0, $\varphi:=\varphi(\bar{x}, I; \cdot)$ and $\hat{\varphi}:=\varphi(\bar{x}, \hat{I}; \cdot)$.

By Theorem 1,

$$N=Z(\varphi; (a, b))\leqq M+S^-(\varphi(a),\dots, \varphi^{(r-1)}(a), (-1)^M)$$

$$-S^+(\varphi(b),\dots, \varphi^{(r-1)}(b), 1)\leqq M+r-m_1-m_2=N.$$

Then,

$$S^-(\varphi(a),\dots, \varphi^{(r-1)}(a),(-1)^M)=r-m_1, \ S^+(\varphi(b),\dots,\varphi^{(r-1)}(b), 1)=m_2.$$

Therefore

$$\varphi^{(j)}(a)\neq 0 \text{ for } j\neq\lambda_1,\dots, \lambda_{m_1} \tag{9}$$

and these non-zero numbers change sign alternatively.

Similar conclusion holds for φ_0 and $\hat{\varphi}$. Particularly,

(10) $$S^-(\hat{\varphi}(a), \ldots, \hat{\varphi}^{(r-1)}(a), (-1)^M) = r - m_1,$$

(11) $$S^-(\varphi_0(a), \ldots, \varphi_0^{(r-1)}(a), (-1)^{M-1}) = r - m_1 - 1.$$

Now it is easy to see that

$$\text{sign } \varphi(t) = \text{sign } \hat{\varphi}(t) = \text{sign } \varphi_0(t) \quad \text{for} \quad a < t < x_1.$$

Since the perfect splines φ, φ_0 and $\hat{\varphi}$ vanish only at $\bar{x}$, the relation above holds in the whole interval (a, b).

Define the function $g(t) := (\varphi(t) - \alpha\varphi_0(t))/(1-\alpha)$, where $\alpha := \varphi^{(k+1)}(a) / \varphi_0^{(k+1)}(a)$. It follows from (9) and (11) that sign $\varphi^{(k+1)}(a) = -\text{sign } \varphi_0^{(k+1)}(a)$ and therefore $\alpha < 0$. Further, on the basis of Theorem B,

(12) $$|\varphi(t)| \leqq |\varphi_0(t)| \quad \text{on} \quad [a, b].$$

Then,

$$|g(t)| = |\varphi(t)| + \frac{|\alpha|}{1 + |\alpha|} |\varphi_0(t) - \varphi(t)|,$$

which yelds $|\varphi(t)| \leqq |g(t)|$.

Now, using again Theorem B with the fact that $g^{(k+1)}(a) = 0$ (and hence, $T(\bar{x}, \hat{I}; g) \equiv \bar{0}$), we get $|g(t)| \leqq |\hat{\varphi}(t)|$. Thus,

(13) $$|\varphi(t)| \leqq |\hat{\varphi}(t)|.$$

Finally, since $\varphi \neq \varphi_0$, the inequality in (12), and consequently in (13) is strict on some subinterval. The proof is complete. The next assertion follows immediately from Theorem 2 on the basis of relation (4).

Corollary 1. *Suppose that $I_1 < I_2$. Then,*

$$R_p(T_1) < R_p(T_2) \quad \textit{for} \quad 1 \leqq p < \infty$$

$$R_p(T_1) \leqq R_p(T_2) \quad \textit{for} \quad p = \infty,$$

where $T_1 = T(\bar{x}, I_1; \cdot)$, $T_2 = T(\bar{x}, I_2; \cdot)$.

The polynomial case of Theorem 2 (i. e., when $N+m_1+m_2=r$) follows in a similar way from Budan-Fourier theorem (see [1] for a simple proof). This particular case was studied first by G. Nikolov [3].

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